Hilbert's 17^{th} Problem

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Question by Hilbert:

Representation of $f(x) \in \mathbb{R}[x_1, \ldots, x_n]$ as sum of squares of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

(False. Counterexample:

$$f(x_1, x_2) = 1 + x_1^2 x_2^4 + x_1^4 x_2^2 - 3x_1^2 x_2^2.)$$

Modified version:

Representation of $f(x) \in \mathbb{R}[x_1, \ldots, x_n]$ as sum of squares of polynomials in $\mathbb{R}(x_1, \ldots, x_n)$.

(Must assume f non-negative i.e. $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.)

General version:

Representation of $f(x) \in R[x_1, \ldots, x_n]$ as sum of squares of polynomials in $R(x_1, \ldots, x_n)$ where $f \geq 0$.

(For what family of R does this holds.)

(Solved by)

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Lemma: Let R be a real closed field containing $\mathbb{Q}$, then $\sum R^2$ is the intersection of the positive cones of all orderings of R.

Theorem: Let R be a real closed field and A and R-algebra of finite type. If there exists an R-algebra homomorphism $\phi : A \to K$ for some real closed extension of R , then there exists an R-algebra homomorphism $\psi : A \to R$.

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Example: The Puiseux series, denoted as $\mathbb{R}(X)$ contains element of the form

$$\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}} \quad \text{with } k \in \mathbb{Z},\ q \in \mathbb{N},\ a_i \in \mathbb{R}$$

It is a non-Archimedean real closed field.

Answer to the Question:

Let R be a real closed field and $f \in R[x_1, \dots, x_n]$. If f is nonnegative on R^n (as a function), then f is a sum of squares in the field of rational functions $R(x_1, \dots, x_n)$.

Proof:

Let R be a real closed field. Suppose on contrary that f cannot be represented as a sum of squares in $R(x_1, \dots, x_n)$. So, there exists an ordering $\preceq$ on $R(x_1, \dots, x_n)$ such that $f \preceq 0$ i.e f is negative with respect to the ordering. Consider the map

$$\phi : \frac{R[x_1, \dots, x_n, T]}{(fT^2+1)} \longrightarrow \overline{R(x_1, \dots, x_n)}$$

$$g(x_1, \dots, x_n, T) \longmapsto g(x_1, \dots, x_n, 1)$$

We show that it is an R-algebra homomorphism. Let $g, h \in \frac{R[X,T]}{(fT^2+1)}$ and $r \in R$, then

$$\begin{aligned} \phi(g+h) &= (g+h)(X,1) \\ &= g(X,1) + h(X,1) \\ &= \phi(g) + \phi(h) \end{aligned}$$

and

$$\begin{aligned} \phi(rg) &= (rg)(X,1) \\ &= rg(X,1) \\ &= r\phi(g). \end{aligned}$$

By Artin-Lang Homomorphism Theorem, there exists an induced R-algebra homomorphism

$$\psi : \frac{R[x_1, \dots, x_n][T]}{(fT^2+1)} \longrightarrow R$$

As 0 maps to 0 in such homomorphism, so $\psi(fT^2+1) = 0$. This implies, $\exists\, (y_1, \dots, y_n) \in R^n$ such that

$$f(y_1, \dots, y_n) * 1^2 + 1 = 0$$

$$f(y_1, \dots, y_n) = -1$$

which is a contradiction as $f \geq 0$ (as a function) on R^n.

Further Development

Q) If $f \in R[x_1, \ldots, x_n]$ and $f \geq 0$, then $f = f_1^2 + \ldots, f_r^2$ for $f_i \in R(x_1, \ldots, x_n)$. Is there any upperbound on r?

Answer

Let R be a real closed field and let $f \in R(x_1, \ldots, x_n)$ is positive definite then there exists $f_1, \ldots, f_{2^n} \in R(x_1, \ldots, x_n)$ such that

$$f = f_1^2 + \cdots + f_{2^n}^2$$

Open Problems

Q) Let K be an arbitrarily field and $f \in K(x)$. Does $f \geq 0$ implies f can be represented as sum of squares in $K(x)$?

Q) If the above holds, is there any bound to the number of squares needed?

References

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3. *Mathematical Development arising from Hilbert Problems*, Proceeding of Symposia in Pure Mathematics.