Generalizing and Classifying Irreducible Numerical Monoids

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We assume $\mathbb{N} = \{0, 1, 2, 3, \rightarrow\}$ throughout the talk.

Definition

A subset $\mathcal{S} \subseteq \mathbb{N}$ is a numerical semigroup if

- $0 \in \mathcal{S}$.
- If $a, b \in \mathcal{S}$ then $a + b \in \mathcal{S}$.
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Example

Let $\mathcal{S} = \{0, 3, 5, 6, 8, 9, 10, \rightarrow\} = \langle 3, 5 \rangle$.

Invariants

Let $\mathcal{S}$ be a numerical semigroup.

- Multiplicity $\mathtt{m}(\mathcal{S})$ is the smallest non-zero number in $\mathcal{S}$.
- Gap set $\mathtt{G}(\mathcal{S})$ is the set of elements of the complement of $\mathcal{S}$ in $\mathbb{Z}_{\geq 0}$. Genus $\mathtt{g}(\mathcal{S})$ is the cardinality of $\mathtt{G}(\mathcal{S})$.

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- Frobenius element $\mathtt{F}(\mathcal{S})$ is the largest number in the gap set $\mathtt{N}(\mathcal{S})$.
- Conductor $\mathtt{c}(\mathcal{S}) = \mathtt{F}(\mathcal{S}) + 1$.
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- Sporadic elements $\mathtt{N}(\mathcal{S}) := \{x \in \mathcal{S} : x < \mathtt{F}(\mathcal{S})\}$. We denote $\mathtt{n}(\mathcal{S}) = |\mathtt{N}(\mathcal{S})|$.
- Minimal generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$.

Example

Let $\mathcal{S} = \{0, 3, 6, 8, 9, 10, \rightarrow\} = \langle 3, 8, 10 \rangle$.

- $\mathtt{m}(\mathcal{S}) = 3$.
- $\mathtt{G}(\mathcal{S}) = \{1, 2, 4, 5, 7\}$ and $\mathtt{g}(\mathcal{S}) = 5$.

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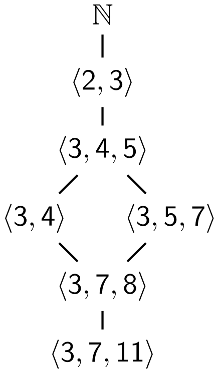

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Let $\mathcal{S}$ be a numerical semigroup. Let $\mathrm{g}(\mathcal{S})$ denote the genus of $\mathcal{S}$.

- We say that S is *symmetric* if $\mathrm{g}(\mathcal{S}) = \frac{1+\mathtt{F}(\mathcal{S})}{2}$.
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We have $\mathtt{G}(\mathcal{S}) = \{1, 2, 4, 5, 8\}$ so, $\mathrm{g}(\mathcal{S}) = 5$. Note $\mathtt{F}(\mathcal{S}) = 8$. So, $\frac{8+2}{2} = 5$ implies $\mathcal{S}$ is Pseudo-symmetric.

Theorem

Let $\mathcal{S}$ be an irreducible numerical semigroup. Then $\mathcal{S}$ is either symmetric or pseudo-symmetric. Moreover, every symmetric or pseudo-symmetric semigroup are irreducible.

Applications

- Let $\mathcal{S}$ be a numerical semigroup. Let $\mathbb{K}$ be algebraically closed and define $\mathbb{K}[\mathcal{S}] = \oplus_{s \in \mathcal{S}} \mathbb{K} t^s$. Consider the ring $\mathbb{K}[[\mathcal{S}]]$.
  [Kun70] showed that $\mathbb{K}[[\mathcal{S}]]$ is a Gorenstein ring if and only if $\mathcal{S}$ is symmetric.

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  [Kun70] showed that $\mathbb{K}[[\mathcal{S}]]$ is a Gorenstein ring if and only if $\mathcal{S}$ is symmetric.
  Remark: A Noetherian ring R is Gorenstein if R has finite injective dimension as an R-module.

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Unipotent Numerical Semigroups

Let

$$\mathbf{U}(n, \mathbb{N}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{ij}\}_{1<i<j<n} \in \mathbb{N} \right\}$$

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We fix a finitely generated monoid $\mathbf{G} \subseteq \mathbf{U}(n,\mathbb{N})$. A subset $\mathcal{S} \subseteq \mathbf{G}$ is a *unipotent numerical semigroup* if

- $\mathbf{1}_n \in \mathcal{S}$.
- If $A, B \in \mathcal{S}$ then $AB \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbf{G}$ is finite.

For Ease

Let us fix $\mathbf{G} = \mathbf{P}(n, \mathbb{N})$ where

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We can simply write an elements of $\mathbf{P}(n, \mathbb{N})$ as $(a_1, \cdots, a_{n-1})$ where $a_i \in \mathbb{N}$.

Example (k-th Fundamental monoid)

Let

$$\mathbf{P}_k(n) := \{(x_j)_{1<j\leq n-1} \in \mathbf{P}(n,\mathbb{N}) : \max_j x_j \geq k\}.$$

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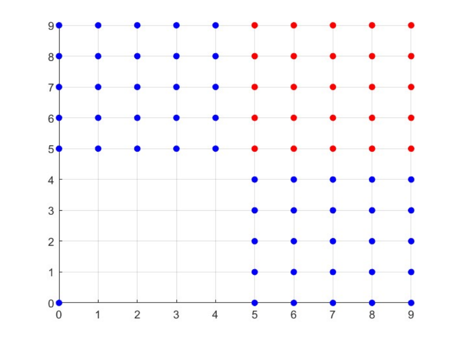

Figure: This is $\mathbf{P}_5(3)$

Example

Let $\mathcal{S} \subseteq \mathbf{P}(3)$ and consider $\mathcal{S}$ plotted as

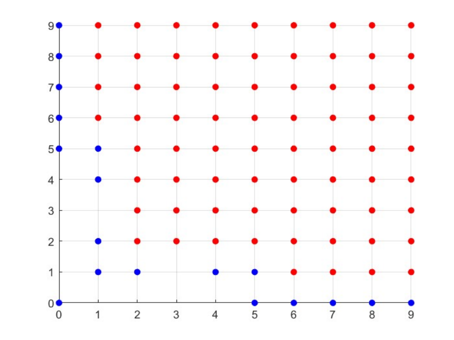

Figure: $\mathcal{S} = \langle (1,1),(2,1),(1,2),(4,1),(1,4),\mathbf{P}_5 \rangle$

Notation

1. From now on, $\mathbf{G}$ denotes $\mathbf{P}(n, \mathbb{N})$.
2. We let $\mathbf{G}_k$ denote the corresponding k-th Fundamental monoid $\mathbf{P}_k(n, \mathbb{N})$.

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2. We let $\mathbf{G}_k$ denote the corresponding k-th Fundamental monoid $\mathbf{P}_k(n, \mathbb{N})$.
3. An asterisk on a set denotes the set minus the identity element e.g. $\mathbf{G}^* = \mathbf{G} \backslash \mathbf{1}_n$ where $\mathbf{1}_n$ denote the $n \times n$ identity matrix.
4. We will denote an arbitrary Unipotent Numerical Monoid in $\mathbf{G}$ by $\mathcal{S}$.

Invariants

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$.

- Gap set $\mathtt{G}(\mathcal{S})$ is the set of elements of the complement of $\mathcal{S}$ in $\mathbf{G}$. Genus $\mathtt{g}(\mathcal{S}) = |\mathtt{G}(\mathcal{S})|$.
- Generating number $\mathtt{r}(\mathcal{S}) = \min\{k \in \mathbb{N} : \mathbf{G}_k \subseteq \mathcal{S}\}$.

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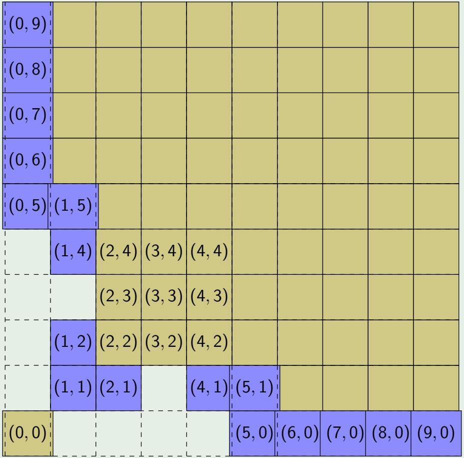
- Sporadic elements $\mathtt{N}(\mathcal{S}) := \mathcal{S} \backslash \mathbf{G}_{\mathtt{r}(\mathcal{S})}$ and $\mathtt{n}(\mathcal{S}) = |\mathtt{N}(\mathcal{S})|$.
- Minimal generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$.

Example

Let $\mathcal{S} = \langle (1,1), (1,2), (1,4), (2,1), (4,1) \rangle \sqcup \mathbf{G}_5$ in $\mathbf{G} = \mathbf{P}(3, \mathbb{N})$.



$\mathtt{r}(\mathcal{S}) = 5, \quad \mathtt{g}(\mathcal{S}) = 10, \quad \mathtt{e}(\mathcal{S}) = 17, \quad \mathtt{n}(\mathcal{S}) = 15$

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. The *Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{F}(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } A\mathbf{G}^* \subseteq \mathcal{S}\}.$$

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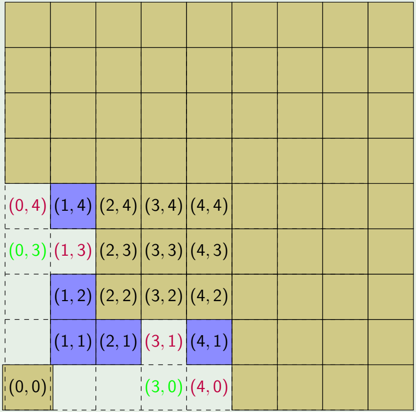
$$\mathrm{PF}(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } A\mathcal{S}^* \subseteq \mathcal{S}\}.$$

Definition

A unipotent numerical monoid $\mathcal{S}$ in $\mathbf{G}$ is said to be irreducible if it cannot be expressed as the intersection of two distinct unipotent numerical monoids properly containing $\mathcal{S}$.

Example (Not Irreducible)

Let $\mathcal{S} = \langle (1,1),(1,2),(1,4),(2,1),(4,1) \rangle \sqcup \mathbf{G}_5$ in $\mathbf{G}$.



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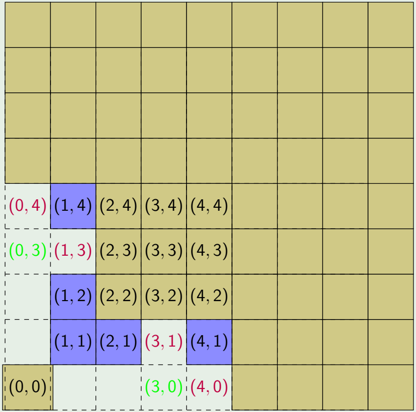
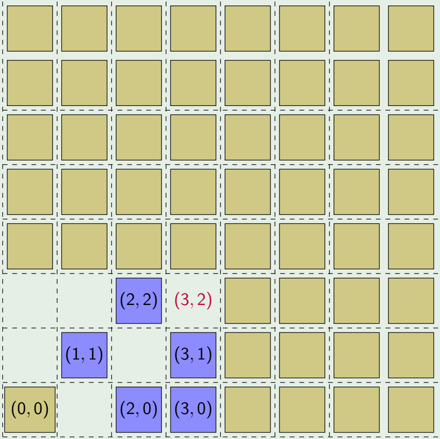


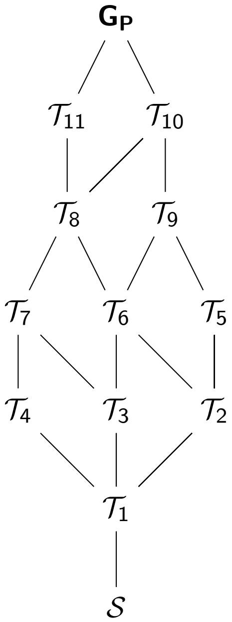
$\mathtt{F}(\mathcal{S}) = \{(0,4),(1,3),(3,1),(4,0)\}$, $\quad \mathtt{PF}(\mathcal{S}) = \mathtt{F}(\mathcal{S}) \cup \{(0,3),(3,0)\}$

Example (Irreducible)

Let $\mathcal{S} = \langle (1,1),(2,0),(2,2),(3,0),(3,1)\rangle \sqcup \mathbf{G}_4$ in $\mathbf{G}$.



$\mathrm{F}(\mathcal{S}) = \{(3,2)\}, \quad \mathrm{PF}(\mathcal{S}) = \mathrm{F}(\mathcal{S})$



$\mathbf{G_P}$
$\mathcal{T}_{11}$
$\mathcal{T}_{10}$
$\mathcal{T}_8$
$\mathcal{T}_9$
$\mathcal{T}_7$
$\mathcal{T}_6$
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- Then $\mathcal{S}$ is called *symmetric* if for every $A \in \mathbf{G} \backslash \mathcal{S}$, we have $\mathtt{F}(\mathcal{S}) \cap (A\mathcal{S}) \neq \emptyset$.

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- Then $\mathcal{S}$ is called *symmetric* if for every $A \in \mathbf{G} \backslash \mathcal{S}$, we have $\mathtt{F}(\mathcal{S}) \cap (A\mathcal{S}) \neq \emptyset$.
- Then $\mathcal{S}$ is called *pseudo-symmetric* if for every $A \in \mathbf{G} \backslash \mathcal{S}$, we have at least one of the following 2 cases:
  1. We have $A^2 \in \mathtt{F}(\mathcal{S})$.
  2. We have $\mathtt{F}(\mathcal{S}) \cap (A\mathcal{S}) \neq \emptyset$.

Symmetric Example

Example

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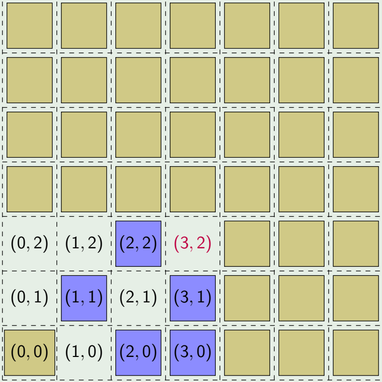

Pseudo-Symmetric Example

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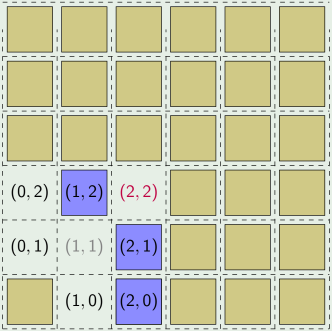

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Let $\mathcal{S}$ be a unipotent numerical monoid in **G**. The following statements are equivalent.

- $\mathcal{S}$ is irreducible.
- $\mathcal{S}$ is maximal with respect to set inclusion in the set of unipotent numerical monoid $\mathcal{S}$ such that $\mathtt{F}(\mathcal{S}) \cap \mathcal{T} = \emptyset$.

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Furthermore, if $|\mathtt{F}(\mathcal{S})| = 1$, then we can add the following equivalent statement to the above list.

- $\mathcal{S}$ is maximal with respect to set inclusion in the set of unipotent numerical monoid $\mathcal{S}$ such that $\mathtt{F}(\mathcal{S}) = \mathtt{F}(\mathcal{T})$.

Theorem (Can, S. 23)

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. If $|\mathtt{F}(\mathcal{S})| = 1$ and for every $A \in \mathbf{G} \backslash \mathcal{S}$, we have $\mathtt{F}(\mathcal{S}) \cap A\mathcal{S} \neq \emptyset$ then $\mathcal{S}$ is irreducible.

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Remark: This shows that the condition of irreducibility in symmetricity can be dropped and be replaced by $|\mathtt{F}(\mathcal{S})| = 1$.

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. We define $\mathcal{S}$ to be symmetric if $|\mathtt{F}(\mathcal{S})| = 1$ and for every $A \in \mathbf{G}\backslash\mathcal{S}$, we have $\mathtt{F}(\mathcal{S}) \cap A\mathcal{S} \neq \emptyset$.

Main Result

Theorem (Can, S. 23)

Let $\mathcal{S}$ be a unipotent numerical monoid. If $\mathcal{S}$ is irreducible, then $\mathcal{S}$ is either symmetric or pseudo-symmetric.

Future directions & problems

1. Connection to commutative algebra.
2. Characterize irreducibility with respect to the set of divisors

$$\mathbf{D}(X) := \{A \in \mathcal{S} : A \leq_{\mathcal{S},t} X\}.$$

3. Derive connection to Weierstrass semigroup of multiple points on a curve X.
4. Connection to algebraic coding theory.

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4. Connection to algebraic coding theory.
5. We look forward to generalizing it to linear algebraic groups. We know that $\mathbf{G} = \mathbf{R} \ltimes \mathbf{U}(n)$.

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THANK YOU!!!