Math 6051/3051: Recitation 10

Naufil Sakran

Do all of the following problems.

(1) Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers.

- $f(x) = x^{17} \sin(x) - e^x \cos(3x)$ on $[0, \pi]$,
  Sol:
  Since $f(x)$ is continuous on the closed interval $[0, \pi]$, it implies $f(x)$ is uniformly continuous on $[0, \pi]$

- $f(x) = \sin \frac{1}{x^2}$ on $(0, 1]$,
  Sol:
  Consider the sequence $x_n = \frac{2}{\sqrt{(2n+1)\pi}}$. Now since $x_n \to 0$, it implies for all $\delta > 0$ there exists N such that for all $n, m > N$, we have $|x_n - x_m| < \delta$. But then $|f(x_n) - f(x_m)| = |(-1)^n - (-1)^m|$, which implies that f is not uniformly continuous.

- $f(x) = x^2 \sin \frac{1}{x}$ on $(0, 1]$.
  Sol:

$$-\lim_{x\to 0} x^2 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le \lim_{x\to 0} x^2$$
$$0 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le 0$$

  which implies $\lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$. Since the function can be extended to a continuous function on $[0, 1]$, this implies f is uniform on $(0, 1]$.

- $f(x) = \frac{1}{x-3}$ on $(4, \infty)$.
  Sol:
  Let $\epsilon > 0$. Now, for $x, y \in (4, \infty)$, we have

$$\left|\frac{1}{x-3} - \frac{1}{y-3}\right| = \left|\frac{y-x}{(x-3)(y_3)}\right|$$
$$= \frac{1}{(x-3)(y-3)}|x-y|$$

  Now since $\frac{1}{(x-3)(y-3)} \le 1$ for $x, y \in (4, \infty)$, we take $|x - y| < \delta = \epsilon$ and have $|f(x) - f(y)| < \epsilon$. So, f is uniformly continuous.

(2) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.
Sol:
Suppose on the contrary that f is not bounded. So, there exists a sequence x_n in S such that $f(x_n)$ diverges. By Weierstrass/Bolzano, there exits a convergent subsequence(x_{n_k}) in (x_n). Let $x_{n_k} \to x$. But then since f is uniform, there exists an extension $\tilde{f}$ of f on $\{S \cup \{x\}\}$ such that $f(x_{n_k}) \to \tilde{f}(x)$. In particular, $(f_{x_{n_k}})$ is bounded implying that $(f(x_n))$ is bounded, a contradiction.

(3) Let $f(x) = \sqrt{x}$ on $(0,1]$. Show that $\frac{df(x)}{dx}$ is unbounded on $(0,1]$ but f is nevertheless uniformly continuous on $(0,1]$.

Sol:

$f(x)$ is uniformly continuous on $(0,1]$ since f can be extended to a continuous function on $[0,1]$ by putting $f(0) = 0$. Now, $\frac{d\,f(x)}{dx} = \frac{1}{2\sqrt{x}}$. Choosing the sequence $x_n = \frac{1}{2n^2}$, we have $\frac{d\,f(x_n)}{dx}$ unbounded.

(4) Let f be a continuous function on $[a.b]$. Show that the function f^* defined as

$$f^*(x) = \sup\{f(y) : a \leq y \leq x\},$$

for $x \in [a,b]$, is an increasing continuous function on $[a,b]$.

Sol:

The function f^* is increasing as for any $x, y \in [a,b]$ such that $x \leq y$, we have

$$\sup\{f(t) \,:\, a \leq t \leq x\} \leq \sup\{f(t) \,:\, a \leq t \leq y\},$$

implying that f^* is increasing on $[a,b]$. Now we show f^* is continuous on $[a,b]$. First of all, we know that f is uniformly continuous on $[a,b]$ (*by theorem in the class*). Now let $\epsilon > 0$. There exists $\delta > 0$ such that for any $x, y \in [a,b]$, whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \frac{\epsilon}{3}$. Suppose $x \leq y$, then there exists $s, t \in [a,b]$ such that $f*(x) = s$ and $f*(y) = t$. We have the direct inequality $s \leq x \leq t \leq y$. Now,

$$\begin{aligned} |f^*(x) - f^*(y)| &= |f(s) - f(t)| \\ &\leq |f(s) - f(x)| + |f(x) - f(y)| + |f(y) - f(t)| \end{aligned}$$

Note that, by uniform continuity, $|f(x) + f(y)| < \epsilon/3$ and $|f(y) - f(t)| < \epsilon/3$. Furthermore, observe that $|f(s) - f(x)| \leq |f(t) - f(x)|$. Also, by uniform continuity, we have $|f(t) - f(x)| < \epsilon/3$. Thus,

$$|f^*(x) - f^*(y)| = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence, $f^*(x)$ is continuous on $[a,b]$.