Math 6051/3051: Recitation 5

Naufil Sakran

Do all **three** of the following problems.

(1) (**Ratio Test**) Assume all $s_n \neq 0$ and that the limit $L = \lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$.
(*Hint: Select* a *so that* $L < a < 1$ *and obtain* N *so that* $|s_{n+1}| < a|s_n|$ *for* $n \geq N$*. Then show* $|s_n| < a^{n-N}|s_N|$ *for* $n > N$*.*)

Sol:

Let $L < \alpha < 1$ and consider $\epsilon < \alpha - L$. By definition of $L = \lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{s_{n+1}}{s_n} - L| < \epsilon$. But then

$$|\frac{s_{n+1}}{s_n} - L| < \epsilon < \alpha - L$$
$$-(\alpha - L) < \frac{s_{n+1}}{s_n} - L < \alpha - L$$
$$-(\alpha - L) + L < \frac{s_{n+1}}{s_n} < \alpha$$
$$0 < \left|\frac{s_{n+1}}{s_n}\right| < \alpha$$

So, $|s_{n+1}| < \alpha|s_n|$ for all $n \geq N$. So, for any $m > N$, we have

$$|s_m| < \alpha|s_{m-1}|$$
$$|s_m| < \alpha^2|s_{m-2}|$$
$$< \cdots$$
$$|s_m| < \alpha^{m-N}|s_N|.$$

Thus, for $m > N$, we have $|s_m| < \alpha^{m-N}|s_N|$. Applying limit on both sides with respect to m, we get

$$\lim_{m\longrightarrow\infty} |s_m| < \lim_{m\longrightarrow\infty} \alpha^{m-N}|s_N|$$
$$0 \leq \lim_{m\longrightarrow\infty} |s_m| = |s_N| \lim_{m\longrightarrow\infty} \alpha^{m-N}.$$

But as $\alpha < 1$, we have $\lim_{m\to\infty} \alpha^{m-N} = 0$. So,

$$0 \leq \lim_{m\to\infty} s_m \leq 0$$

implies $\lim_{m\to\infty} s_m = 0$.

(b) Show that if $L > 1$, then $\lim s_n = +\infty$. (*Hint: Apply (a) to the sequence* $t_n = \frac{1}{|s_n|}$ *and use the fact* $\lim |s_n| = \infty$ *if and only if* $\lim \frac{1}{|s_n|}$*=0*)

Sol:

Let $t_n = \frac{1}{|s_n|}$. Then since $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L > 1$, we have $\lim_{n\to\infty} \left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty} \left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{L} < 1$. By above, this implies $\lim t_n = 0$. Using the fact that $\lim |x_n| = \infty$ if and only if $\lim \frac{1}{|x_n|}$=0, we have $\lim_{n\to\infty} |s_n| = \infty$.

(c) Use this to show that if $a > 0$, then

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

Sol:

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} \right| &= \lim_{n\to\infty} \left| \frac{a}{n+1} \right| \\ &= 0\end{aligned}$$

By the Ratio Test, this implies

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

(2) Which of the following sequences are increasing? decreasing? bounded?

(a) $\frac{1}{n}$.
Sol:

Decrasing and bounded.

(b) $\frac{(-1)^n}{n^2}$.
Sol:

Bounded.

(c) $\sin\left(\frac{n\pi}{7}\right)$.
Sol:

Bounded.

(d) $\frac{n}{3^n}$
Sol:

Decreasing after some point and bounded.

(3) Let (s_n) be a sequence
(a) Suppose

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.
Sol:

We know that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$. Now for any $m > n$, we have

$$\begin{aligned}
|s_m - s_n| &< |s_m - s_{m-1} + s_{m-1} - s_{m-2} + s_{m-2} - \cdots + s_{n+1} - s_n| \\
&\le |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+2} - s_{n+1}| + |s_{n+1} - s_n| \\
&< 2^{-(m-1)} + 2^{-(m-2)} + \cdots + 2^{-(n+1)} + 2^{-n} \\
&= 2^{-n}(2^{-(m-n-1)} + 2^{-(m-n-2)} + \cdots + 2^{-1} + 1) \\
&\le 2^{-n} \cdot \frac{1}{1-1/2} \qquad \blacksquare \\
&= 2^{-n+1}.
\end{aligned}$$

In conclusion, for any $m > n$, we have

$$|s_m - s_n| < 2^{-n+1}.$$

Now for any $\epsilon > 0$, choose $N \in N$ such that for all $n \ge N$, we have

$$2^{-n+1} < \epsilon.$$

So, for any $m > n \ge N$, we have

$$|s_m - s_n| < 2^{-n+1} < \epsilon.$$

Since, $\epsilon > 0$ was arbitrary, we have that (s_n) is a Cauchy sequence.

(b) Is the result in part (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$. If it is true, prove it, and if it is false, give a counterexample.

Sol:

This is not necessarily true because we will have problem at the $\blacksquare$ step i.e. our sequence will not be bounded by a convergent sequence.

A counterexample would be to consider the sequence

$$s_n = \sum_{k=1}^{n} \frac{1}{k}$$

which is the n^{th} partial sum of the harmonic series. Then,

$$\begin{aligned}
|s_{n+1} - s_n| &= \left| \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right| \\
&= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) \right| \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

So, we have $|s_{n+1} - s_n| < \frac{1}{n}$. But then the sequence s_n does not converge as

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$