Math 6051/3051: Recitation 6

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Do all of the following problems.

(1) Show that Cauchy sequences are bounded.

Sol: Let (x_n) be a Cauchy sequence and let $\epsilon = 1$. Then there exists N such that for all $n, m \geq N$, we have

$$|x_n - x_m| < \epsilon = 1.$$

In particular,

$$|x_{N+k} - x_N| < 1$$

for all $k \geq 1$. Note that

$$\begin{aligned} -1 < & x_{N+k} - x_N < 1 \\ & = -1 + x_N < \qquad x_{N+k} < 1 + x_N \\ & = 0 \leq |x_{N+k}| < 1 + |x_N|. \end{aligned}$$

Thus, letting $M = \max\{|x_1|, |x_2|, \cdots, |x_{N-1}|, 1 + |x_N|\}$, we have $|x_n| \leq M$ for all $n \geq 1$. Hence, (x_n) is bounded.

(2) Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) \cdot t_n$. Use induction to show that

$$t_n = \frac{n+1}{2n},$$

and then compute $\lim_{n\to\infty} t_n$.

Sol: For $n = 1$, we have $t_2 = (1 - \frac{1}{2^2}) * 1 = \frac{3}{4} = \frac{2+1}{2*2}$. Suppose it holds for n, i.e. $t_n = \frac{n+1}{2n}$. Now,

$$\begin{aligned} t_{n+1} &= \left(1 - \frac{1}{(n+1)^2}\right) \cdot t_n \\ &= \left(1 - \frac{1}{(n+1)^2}\right) \cdot \frac{n+1}{2n} \\ &= \frac{(n+1)^2 - 1}{(n+1)^2} \cdot \frac{n+1}{2n} \\ &= \frac{n^2 + 2n}{2n(n+1)} \\ &= \frac{(n+1) + 1}{2(n+1)} \end{aligned}$$

Hence, proved.

(3) Let S be a bounded set. Prove there is an increasing sequence (s_n) of points in S such that

$$\lim s_n = \sup S.$$

Sol:

Let S be a non-empty bounded set. We construct an increasing sequence, converging to $\sup S$. If $\sup S \in S$, then we can take the constant sequence $(\sup S, \sup S, \cdots$, and we are done. Now suppose $\sup S \notin S$. Let $\epsilon = 1$. Then by definition, there exists $x \in S$ such that $\sup S - 1 \leq x < \sup S$. Let $x_1 = x$. Now to pick x_2, choose $\epsilon = \sup S - x_1$. Then there exists $y \in S$ such that $\sup S - \epsilon \leq y < \sup S$ and we take $x_2 = y$. By construction, we have $x_1 \leq x_2$. Now suppose we have constructed an increasing sequence up to the nth term. To pick an element for x_{n+1}, we take $\epsilon = \sup S - x_n$. Then there exists $y \in S$ such that $\sup S - \epsilon \leq y < \sup S$ and we take $x_{n+1} = y$. Clearly $x_n \leq x_{n+1}$. Thus, we have constructed an increasing sequence (x_n). It remains to show that our constructed sequence (x_n) converges to $\sup S$. Let $\epsilon > 0$. Then there exists $t \in S$ such that $\sup S - \epsilon < t < \sup S$. By construction of (x_n), there exists N such that for all $n > N$, we have $t < x_n$. Hence, taking $\epsilon \to 0$, we have $x_n \to \sup S$.

(4) Prove that $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$. Given an example of such a sequence.
Sol:

Let $\limsup |s_n| = 0$. This implies for $\epsilon > 0$, there exists N such that $\sup\{|s_{N+1}|, |s_{N+2}|, \cdots\} < \epsilon$. But then this implies $|s_n| < \epsilon$, implying that $|s_n| \to 0$. Conversely, let $\lim |s_n| = 0$. Then for $\epsilon > 0$, there exists N such that for all $n > N$, we have $|s_n| < \epsilon$. Equivalently, this means $\sup\{|s_{N+1}|, |s_{N+2}|, \cdots\} < \epsilon$, implying that $\limsup |s_n| = 0$.

(5) Prove that (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.
Sol:

Suppose (s_n) is bounded then there exists M such that $|s_n| < M$ for all $n \geq 1$. Thus, $\limsup |s_n| < M < +\infty$. Conversely, if $\limsup |s_n| < +\infty$, then for every $n \geq 1$, $\sup\{s_n, s_{n+1}, \cdots\} < +\infty$. In particular, taking $n = 1$, we have (s_n) is bounded.