Math 6051/3051: Recitation 7

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Do all of the following problems.

(1) Let $f(x)$ be a continuous real-valued function with domain $(a, b) \subseteq \mathbb{R}$. Show that if $f(r) = 0$ for every rational number $r \in (a, b)$, then $f(x) = 0$ for all $x \in (a, b)$.

Sol:

Suppose $f(x)$ be a continuous real-valued function with domain (a, b) such that $f(r) = 0$ for all rational numbers $r \in (a, b)$. Now for any irrational $x \in (a, b)$, we know that there exists a sequence of rational numbers r_n converging to x i.e. $r_n \to x$. Then since f is continuous, we have

$$\lim f(r_n) = f(\lim r_n) = f(x).$$

But then as $f(r_n) = 0$ for all n, we have $f(x) = 0$. Since, x was an arbitrary irrational number, so $f(x) = 0$ for any $x \in (a, b)$.

(2) Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$. Use this to conclude that every polynomial function $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

Sol:

Let (x_n) be a sequence converging to x_0. To show $f(x) = x^m$ is continuous, we need to show that (x_n^m) converges to x_0^m. We know that

$$|x_n^m - x_0^m| = |x_n - x_0||x_n^{m-1} + x_n^{m-2}x_0 + \cdots x_n x_0^{m-2} + x_0^{m-1}|$$

Now since, (x_n) is a convergent sequence, it is bounded, say by M. Thus for $\epsilon > 0$, choose N such that for all $n > N$ we have $|x_n - x_0| < \frac{\epsilon}{mM^m}$. Then

$$\begin{aligned}
|x_n^m - x_0^m| &= |x_n - x_0||x_n^{m-1} + x_n^{m-2}x_0 + \cdots x_n x_0^{m-2} + x_0^{m-1}| \\
&\leq |x_n - x_0|(|M^m| + |M^m| + \cdots + |M^m|) \\
&< \frac{\epsilon}{mM^m} \cdot mM^m = \epsilon.
\end{aligned}$$

So, (x_n^m) converges to x_0^m and thus $f(x)$ is continuous. By the last question of the recitation, we have that the sum and product of continuous function is continuous, This implies every polynomial function $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

(3) A *rational function* is a function f of the form $\frac{p}{q}$ where p and q are polynomial functions. The domain of f is $\{x \in \mathbb{R} \mid q(x) \neq 0\}$. Use the previous exercise to prove that every rational function is continuous.

Sol:

Again by citing the last question of the recitation, product of continuous functions is continuous so $\frac{1}{p(x)}$ is also continuous for those x, for which $p(x) \neq 0$. Thus, every rational function is continuous.

(4) Consider the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Show that f is discontinuous for every $x \in \mathbb{R}$.

Sol:

Let $x \in \mathbb{R}$. if x is irrational, then there exists a sequence of rational numbers r_n converging to x. But then

$$\lim f(r_n) = 1 \neq 0 = f(x).$$

So, f is discontinuous at irrational numbers. Similarly, we know that for any rational number r, t with $r < t$, there exists an irrational number x such that $r < x < t$. This gives us a sequence of irrational numbers q_n converging to any rational number r. So,

$$\lim f(q_n) = 0 \neq 1 = f(x).$$

Thus, f is discontinuous at every $x \in \mathbb{R}$.

(5) Let f and g be two continuous functions on $\mathbb{R}$. Show that $f + g$ and fg are continuous.
Sol:

Let f and g be continuous function and let (x_n) be a sequence converging to x. Then

$$\lim(f+g)(x_n) = \lim f(x_n) + \lim g(x_n) = f(x) + g(x) = (f+g)(x),$$

which implies $f + g$ is continuous. Similarly,

$$\lim(fg)(x_n) = (\lim f(x_n))(\lim g(x_n)) = f(x)g(x) = (fg)(x),$$

which implies fg is continuous.

A better way to do this is as follows:

Let $x_0 \in domain(f) \cap domain(g)$. Let $\epsilon > 0$. Then there exists δ_1 and δ_2 such that for every $x \in dom(f) \cap dom(g)$, if $|x - x_0| \leq \delta_1$ then $|f(x) - f(x_0)| < \epsilon/2$ and if $|x - x_0| \leq \delta_2$ then $|g(x) - g(x_0)| < \epsilon/2$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then for $|x - x_0| < \delta$, we have

$$\begin{aligned} |(f+g)(x) - (f+g)(x_0)| &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This implies $f + g$ is continuous at x_0. For the next part, observe that

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\ &\leq |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| \end{aligned}$$

Now let $\epsilon > 0$. There exists δ_1 such that for $|x - x_0| < \delta_1$, we have $|f(x) - f(x_0)| < \frac{\epsilon}{2|g(x_0)|}$. Similarly, there exists δ_2 such that whenever $|x - x_0| < \delta_2$, we have $|f(x) - f(x_0)| < \epsilon$. This implies $|f(x)| \leq \epsilon + |f(x_0)|$ whenever $|x - x_0| < \delta_2$. Finally, there exists δ_3 such that whenever $|x - x_0| < \delta_3$, we have $|g(x) - g(x_0)| < \frac{\epsilon}{2(\epsilon + |f(x_0)|)}$. Thus, taking $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\ &\leq |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| \\ &< (\epsilon + |f(x_0)|) \cdot \frac{\epsilon}{2(\epsilon + |f(x_0)|)} + |g(x_0)| \cdot \frac{\epsilon}{2|g(x_0)|} \\ &= \epsilon. \end{aligned}$$

Thus, fg is continuous.