Math 6051/3051: Recitation 8

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Do all of the following problems.

(1) Determine whether the following series converges or diverges. If it converge, show why.

- $\sum \frac{n!}{n^4+3}$.
  Sol:
  Diverges as $\frac{n!}{n^4+3} \to \infty$
- $\sum_{n=2}^{\infty} \frac{1}{\log n}$.
  Sol:
  Diverges as $\frac{1}{n} < \frac{1}{\log n}$, implying that $\sum \frac{1}{n} < \sum \frac{1}{\log n}$ and $\sum \frac{1}{n}$ diverges.
- $\sum \frac{n^2}{n!}$.
  Sol:
  Converges as $\frac{n^2}{n!} \leq \frac{n^2}{n^n} = \frac{1}{n^{n-2}} \leq \frac{1}{n^2}$. implying that $\sum \frac{n^2}{n!} \leq \sum \frac{1}{n^2}$.
- $\sum \frac{1}{\sqrt{n!}}$.
  Sol:
  Converges as $\frac{1}{\sqrt{n!}} \leq \frac{1}{(n!)^{\frac{2}{n}}} \leq \frac{1}{(n^n)^{\frac{2}{n}}} \leq \frac{1}{n^2}$.

(2) Determine which of the following series converge. Justify your answers.

- $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$.
  Sol:
  Since, $\frac{1}{n\log n} \leq \frac{1}{\sqrt{n}\log n}$, we test $\sum \frac{1}{n\log n}$. Using the integral test

$$\begin{aligned}\int_2^{\infty} \frac{1}{x\log x}dx &= \log\log x|_2^{\infty} \\ &= \infty\end{aligned}$$

  So, $\sum \frac{1}{\sqrt{n}\log n}$ diverges.

- $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$.
  Sol:

$$\begin{aligned}\int_2^{\infty} \frac{\log x}{x^2}dx &= \left(-\frac{\log x}{x} + \int \frac{1}{x^2}dx\right|_2^{\infty} \\ &= \lim_{b\to\infty}\left(-\frac{\log x}{x} - \frac{1}{x}dx\right|_2^{b} \\ &= \lim_{b\to\infty}\left(-\frac{\log b}{b} - \frac{1}{b}dx\right) + \frac{\log 2}{2} + \frac{1}{2} \\ &= \frac{\log 2}{2} + \frac{1}{2}\end{aligned}$$

  So, $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converges.

- $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}(\log n)(\log\log n)}$.
  Sol:

$$\begin{aligned}\int_2^\infty \frac{1}{x(\log x)(\log\log x)}dx &= \int_2^\infty \frac{1}{x(\log x)(\log\log x)}dx\\ &= \int_2^\infty \frac{1}{u(\log u)}du\\ &= \log v|_2^\infty\\ &= \infty.\end{aligned}$$

Thus, since $\sum_{n=2}^\infty \frac{1}{n(\log n)(\log\log n)} \leq \sum_{n=2}^\infty \frac{1}{\sqrt{n}(\log n)(\log\log n)}$ and $\sum_{n=2}^\infty \frac{1}{n(\log n)(\log\log n)}$ diverges implies $\sum_{n=2}^\infty \frac{1}{\sqrt{n}(\log n)(\log\log n)}$ diverges.

(3) Prove that the following function is continuous at x_0, using the $\epsilon - \delta$ definition.

$$f(x) = \sqrt{x}, \quad x_0 = 0.$$

Sol:
Let $\epsilon > 0$. Choosing $\delta = \epsilon^2$, we have whenever $|x| < \delta$ implies $|\sqrt{x}| < \epsilon$. So, f is continuous at $x_0 = 0$.

(4) For each non-zero rational number x, write x as $\frac{p}{q}$, where p, q are integers with no common factors and $q > 0$. Define $f(x) = \frac{1}{q}$. Also define $f(0) = 1$ and $f(x) = 0$ for all $x \in \mathbb{R}\backslash\mathbb{Q}$. For instance, $f(x) = 1$ for every integer x and $f(\frac{1}{2}) = (-\frac{15}{2}) = \frac{1}{2}$ etc. Show that f is continuous at each point of $\mathbb{R}\backslash\mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.
Sol:
Let $x \in \mathbb{R}\backslash\mathbb{Q}$. Then writing x in a decimal form, we have

$$x = a_0.a_1a_2a_3a_4\cdots$$

Let (r_n) be a sequence converging to x. Without loss of generality, we can take (r_n) to be a sequence of rational numbers since $f(t) = 0$ for $t \in \mathbb{R}\backslash\mathbb{Q}$. Now consider the sequence (y_k) defined as $y_k = a_0.a_1a_2\cdots a_k$. Since, $r_n \to x$, there exists a subsequence $(t_s) \subseteq (y_k)$ and a large N such that for $(t_s) = (r_n)_{n\geq N}$. Now, since $f(y_k) \to 0 = f(x)$ implies $f(t_s) \to 0 = f(x)$. In particular, $f(r_n) \to 0 = f(x)$. Since, (r_n) was an arbitrary sequence, we have that f is continuous at every irrational point x.
Now let $x_0 \in \mathbb{Q}$ and $0 < \epsilon < f(x_0)$. Now for any $\delta > 0$ and any irrational number y such that $|y - x_0| < \delta$, we have $|f(x_0) - f(y)| = f(x_0) > \epsilon$. Thus, f is discontinuous at every rational point.