Math 6051/3051: Recitation 9

Naufil Sakran

Do all of the following problems.

(1) Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to a number $x \notin S$. Show that there exists an unbounded continuous function on S.
Sol:
Let (x_n) be a sequence in S converging to $x_0 \notin S$. Consider the function $f(y) = \frac{1}{y - x_0}$. We show that f is unbounded. Let M be a large number. Then choose $\delta < \frac{1}{M}$ such that $|x_n - x_0| < \delta$. But then $|f(x_n)| = \frac{1}{|x_n - x_0|} > M$. This implies f is an unbounded continuous function.

(2) Let f and g be continuous function on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.
Sol:
Consider the function $h(x) = f(x) - g(x)$. Clearly h is continuous. Furthermore, $h(a) \geq 0$ and $h(b) \leq 0$. By intermediate value theorem, there exists $x_0 \in [a, b]$ such that $h(x_0) = 0$, which implies $f(x_0) = g(x_0)$.

(3) Suppose f is a real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists x between a and b such that $f(x) = 0$
Sol:
Suppose $f(a) < 0$ and $f(b) > 0$. By intermediate value theorem, there exists $x_0 \in [a, b]$ such that $f(x_0) = 0$. Similarly, if $f(a) > 0$ and $f(b) < 0$. By intermediate value theorem, there exists $x_0 \in [a, b]$ such that $f(x_0) = 0$.

(4) Let

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that $f(x)$ is discontinuous at $x = 0$.
Sol:
Consider the sequence $x_n = \frac{2}{(2n+1)\pi}$. Clearly $x_n \to 0$. But $f(x_n) = \sin \frac{(2n+1)\pi}{2} = (-1)^n$. So, $x_n \to 0$ but $\lim_{n\to\infty} f(x_n) = DNE$. So, f is discontinuous at 0.