Real Root Counting

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1. Introduction
2. Descarte's Law of signs and Budan-Fourier theorem
3. Sturm's Theorem

History

Determining the roots of polynomials, or "solving algebraic equations", is among the oldest problems in Mathematics. However, the elegant and practical notation we use today only developed beginning in the 15th century. Before that, equations were written out in words. Our current formal definition of polynomial in one variable is:

Definition

Let R be a ring *(a well defined "good" Mathematical structure)*. We define the set of polynomials $R[x]$ to be

$$R[x] := \{\sum_{i=0}^{n} a_i x^i \ : \ a_i \in R, \forall\, i \ \text{ and } \ n \in \mathbb{N}\}$$

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Given any $f(x) \in R[x]$*, what can you say about the existence and quantity of solution to* $f(x) = 0$ *where* $x \in A$ *where* A *is a Mathematical structure "friendly" or "compatible" with* R*?*

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Given any $f(x) \in R[x]$, what can you say about the existence and quantity of solution to $f(x) = 0$ where $x \in A$ where A is a Mathematical structure "friendly" or "compatible" with R?

The most general place where x could possibly come from is from all rings containing R.

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STILL VERY HARD TO ANSWER IN GENERAL!!! If we impose some additional structure on R, we might completely answer some of the above problems.

Real closed fields

We will let R be a real closed field and discuss our problem in this case.

Definition (Real closed field)

R is said to be **real closed field** if R is a totally ordered field and $R[i] = \frac{R[x]}{x^2+1}$ is algebraically closed. [2]

Examples

- $\mathbb{R}$ (Of course)
- $\mathbb{Q}$ (Real field but not real closed field)
- $\mathbb{R}_{\mathrm{alg}}$ (Real closure of $\mathbb{Q}$)
- Puiseux series. (Building blocks are of the form $\sum_{k=k_0}^{\infty} c_k X^{k/n}$ where $c_k \in F$ and $k_0, n(\neq 0) \in \mathbb{Z}$)

Objective

Let R be a real closed field. Let $f(x) \in R[x]$. So, f has the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad , a_n \neq 0$$

Our goal is too see whether the coefficients a_i has anything to do with the roots of f or not.

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Definition

Let $a = (a_1, a_2, \ldots, a_n)$ be a sequence in $R\backslash\{0\}$. We define **number of sign variations** $\mathrm{Var}(a)$ **in** a to be

$$\mathrm{Var}(a_0, \ldots, a_p) = \begin{cases} \mathrm{Var}(a_1, \ldots, a_p) + 1 & \text{if } a_0 a_1 < 0 \\ \mathrm{Var}(a_1, \ldots, a_p) & \text{if } a_0 a_1 > 1 \end{cases}$$

If we have sequence containing 0, take the new sequence by removing 0. Also define $\mathrm{Var}(\emptyset) = 0$.

Example

$$\mathrm{Var}(1, -1, 2, 0, 0, 3, 4, -5, -2, 0, 3) = \mathrm{Var}(1, -1, 2, 3, 4, -5, -2, 3) = 4$$

Now for any $f = \sum_{i=0}^{n} a_i x^i \in R[x]$,

$$\mathrm{Var}(f) = \mathrm{Var}(a_0, a_1, \ldots, a_n)$$

Definition

Let $\mathrm{Pos}(f)$ denote the **number of positive solutions** of f.

Descarte's Law of Signs

Theorem

Let R be a real field and $f \in R[x]$ then

1. $Var(f) \geq Pos(f)$.
2. $Var(f) - Pos(f)$ *is even.*

General version

Definition

Let $f = f_0, f_1, \ldots, f_d$ be a sequence of polynomials and let $a \in R \cup \{\pm\infty\}$. The **number of sign variations** of f at a, denoted by $\mathrm{Var}(f; a)$, is

$$\mathrm{Var}(f; a) = \mathrm{Var}(f_0(a), f_1(a), \ldots, f_d(a))$$

For any interval $(a, b] \subset R$,

$$\mathrm{Var}(f; a, b) = \mathrm{Var}(f; b) - \mathrm{Var}(f; a)$$

Budan-Fourier Theorem

Theorem (Budan-Fourier Theorem)

Let R be a real field and $f \in R[x]$ of degree n. Let $\mathrm{Der}(f) = (f, f', \ldots, f^{(n)})$ be the sequence of derivatives of f. Given any $a, b \in R \cup \{\pm\infty\}$

1. $\mathrm{Var}(\mathrm{Der}(f); a, b) \geq \mathrm{num}(f; (a, b])$.
2. $\mathrm{Var}(\mathrm{Der}(f); a, b) - \mathrm{num}(f; (a, b])$ *is even.*

where $\mathrm{num}(f; (a, b])$ denote the number of roots of f in $(a, b]$. [1]

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Definition (Signed Remainder sequence)

Let R be a real field. Let $f, g \in R[x]$, not both 0. The **signed remainder sequence of f and g** is defined as

$$\mathrm{sRem}(f,g) = (\mathrm{sRem}_0(f,g), \mathrm{sRem}_1(f,g), \dots, \mathrm{sRem}_k(f,g))$$

where

$$\mathrm{sRem}_0(f,g) = f, \qquad \mathrm{sRem}_1(f,g) = g,$$

and for $i \geq 1$, if $\mathrm{Rem}(\mathrm{sRem}_{i-1}(f,g), \mathrm{sRem}_i(f,g)) \neq 0$,

$$\mathrm{sRem}_{i+1}(f,g) = -\mathrm{Rem}(\mathrm{sRem}(f,g)_{i-1}, \mathrm{sRem}_i(f,g))$$

where $\mathrm{Rem}(P,Q)$ denote the remainder of P divided by Q for any $P, Q \in R[x]$, not both zero. *(k is such that $\mathrm{sRem}_i(f,g) = 0$ for all $i > k$).*

Theorem (Sturm's Theorem)

Let R be a real field and $f \in R[x]$. Given $a, b \in R \cup \{\pm\infty\}$ that are not roots of f,

$$Var(sRem(f, f'); a, b) = num(f; (a, b))$$

where $num(f; (a, b])$ denote the number of roots of f in (a, b). [1]

Example

Let $f = x^4 - 5x^2 + 4 \in \mathbb{R}[x]$.

$$\mathrm{sRem}_0(f, f') = f = x^4 - 5x^2 + 4 \qquad \mathrm{sRem}_3(f, f') = \frac{18}{5}x$$

$$\mathrm{sRem}_1(f, f') = f' = 4x^3 - 10x \qquad \mathrm{sRem}_4(f, f') = 4$$

$$\mathrm{sRem}_2(f, f') = \frac{5}{2}x^2 - 4$$

$$\mathrm{Var}(\mathrm{sRem}(f, f'); \infty) = \mathrm{Var}(+, +, +, +, +) = 0$$

$$\mathrm{Var}(\mathrm{sRem}(f, f'); -\infty) = \mathrm{Var}(+, -, +, -, +) = 4$$

So, $\mathrm{Var}(\mathrm{sRem}(f, f'); \infty, -\infty) = \mathrm{num}(f; (-\infty, \infty)) = 4$.

[1] Saugata Basu, Richard Pollack, and Marie Françoise. Roy. *Algorithms in Real algebraic geometry*. Vol. 10. Springer Science, 2016.

[2] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*. Vol. 36. Springer Science & Business Media, 2013.

Thank You

For questions, you can email me at nsakran@tulane.edu