Math 1221: Recitation 10 (T)

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(1) Solve the following.

(a) Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to 0$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{2^{n+1}} * \frac{2^n}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n} * \frac{x}{2}\right| \to 0 * \frac{|x|}{2} = 0 < 1$$

So for any value of x, the series converges which implies $R = \infty$.

(b) Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to c < 1$ where $c \neq 0$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{2^{n+1}} * \frac{2^n}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n} * \frac{x}{2}\right| \to c * \frac{|x|}{2} = \frac{c|x|}{2} < 1$$

So $|x| < \frac{2}{c}$ implying that $R = \frac{2}{c}$.

(c) Suppose $\sqrt[n]{|a_n|} \to 1$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n (x-2)^n}{9^n}$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{a_n(x-2)^n}{9^n}\right|} = \lim_{n\to\infty}\sqrt[n]{|a_n|}\frac{|x-2|}{9} \to \frac{|x-2|}{9} < 1$$

So, $\frac{|x-2|}{9} < 1$ implies $|x-2| < 9$ implying $R = 9$.

(2) Solve the following questions. **(Do any two of them)**.

(a) Use partial fractions to find the power series of the function. *(Hint:* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$*)*

$$\frac{7}{(x-10)(x+1)}$$

$$\frac{7}{(x-10)(x+1)} = \frac{7}{11} * \frac{1}{x-10} - \frac{-7}{11} * \frac{1}{x+1} = \frac{7}{11 * 10}\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n + \frac{7}{11}\sum_{n=0}^{\infty}(-x)^n$$

(b) Differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f

$$f(x) = \frac{1}{1+x^8}.$$

$$f(x) = \frac{1}{1+x^8} = \sum_{n=0}^{\infty}(-1)^n x^{8n}$$

$$f'(x) = \sum_{n=0}^{\infty}(-1)^n 8n x^{8n-1}$$

(c) Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

as $\int_0^1 f(t)dt$ where $f(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$ by identifying it as the value of a derivative or integral of geometric series.

Let $f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. So,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

So, $\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1}$ implies $\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \frac{\pi}{4}$.

(3) **(Bonus)** Solve any **two of them.**

(a) Given $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Use term-by-term differentiation or integration to find a power series for the function centered at the given point.

$$f(x) = \ln(1-x^8) \text{ centered at } x = 0.$$

$$\frac{d}{dx}\ln(1-x^8) = \frac{-8x^7}{1-x^8}$$

So,

$$\frac{-8x^7}{1-x^8} = -8x^7 * \frac{1}{1-x^8}$$

$$\frac{-8x^7}{1-x^8} = -8x^7 \sum_{n=0}^{\infty} x^{8n}$$

$$\frac{-8x^7}{1-x^8} = \sum_{n=0}^{\infty} -8x^{8n+7}$$

Now integrating both sides

$$\int \frac{-8x^7}{1-x^8}dx = \int \sum_{n=0}^{\infty} -8x^{8n+7}dx$$

$$\ln(1-x^8) = \sum_{n=0}^{\infty} \frac{-8x^{8n+8}}{8n+8}.$$

(b) Find the Taylor series at $a = \frac{\pi}{2}$ for

$$f(x) = 10\cos x$$

$$f(\frac{\pi}{2}) = 10\cos\frac{\pi}{2} = 0$$
$$f'(\frac{\pi}{2}) = -10\sin\frac{\pi}{2} = -10$$
$$f''(\frac{\pi}{2}) = -10\cos\frac{\pi}{2} = 0$$
$$f'''(\frac{\pi}{2}) = 10\sin\frac{\pi}{2} = 10$$
$$f^{(4)}(\frac{\pi}{2}) = 10\cos\frac{\pi}{2} = 0$$

So,

$$f(x) = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x-\frac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\frac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\frac{\pi}{2})^3 + \cdots$$
$$f(x) = 0 - 10(x-\frac{\pi}{2}) + 0 + \frac{10}{3!}(x-\frac{\pi}{2})^3 - 10\frac{10}{5!}(x-\frac{\pi}{2})^5 + \cdots$$

(c) Find the integral $\int_0^1 \cos(x^2)dx$ in terms of series. As

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

So,

$$\int \cos x^2 dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!(4n+3)} + C$$