The Reimann-Hurwitz Formula

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The final project for Fiber Bundles class
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2. Holomorphic maps between Reimann Surfaces
3. Order, Multiplicity, and Euler number
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Basic Definition

We denote $\mathbb{C}$ to be the complex plane in which the imaginary part is denoted by ι.

Complex structures

Let X be a topological space. A *complex chart* on X is a homeomorphism $\phi : U \to V$, where U is an open set in X and V is an open set in $\mathbb{C}$.

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$$\phi_2 \circ \phi_1^{-1} : \phi(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

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is holomorphic. A *complex atlas* $\mathcal{A}$ on X is a collection of compatible complex charts $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ such that $X \subseteq \bigcup_\alpha U_\alpha$.

Reimann Surface

A *Reimann surface* is a second countable connected Hausdorff topological space X with a complex structure.

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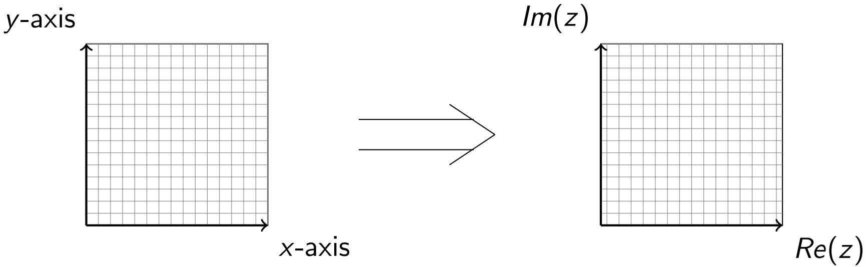
Example

Let $X = \mathbb{R}^2$, and let U be any open subset. Define $\phi_U : U \to V \subseteq \mathbb{C}$ such that $(x, y) \mapsto x + \iota y$. This is a complex chart.



Example (Reimann Sphere $\mathbb{C}^{\infty}$)

Let $S^2 := \{(x, y, w) \mid x^2 + y^2 + w^2 = 1\}$ denote the unit 2-sphere inside $\mathbb{R}^3$. Consider the maps

$$\phi_1 : S^2 - (0,0,1) \to \mathbb{C}, \quad \phi_1(x,y,w) = \frac{x}{1-w} + \iota \frac{y}{1-w}$$
$$\phi_2 : S^2 - (0,0,-1) \to \mathbb{C}, \quad \phi_2(x,y,w) = \frac{x}{1+w} + \iota \frac{y}{1+w}$$

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Then ϕ_1 and ϕ_2 are compatible complex charts as $\phi_2 \circ \phi_1^{-1}(z) = \frac{1}{z}$, which is holomorphic on $\phi_i(S^2 - (0,0,\pm 1)) = \mathbb{C} - \{0\}$. The set $\mathcal{A} = \{\phi_1, \phi_2\}$ forms an atlas of S^2.

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We denote Reimann Sphere by $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$, because if we subject to the chart ϕ_1, we map $(0,0,1)$ to $\{\infty\}$ and if we subject to the chart ϕ_2, we map $(0,0,-1)$ to $\{\infty\}$.

Forming a Reimann surface

1. Start with a set X.
2. Find a countable collection of subsets $\{U_\alpha\}$ covering X.
3. For each α, find a bijection $\phi_\alpha : U_\alpha \to V_\alpha$ where V_α is open in $\mathbb{C}$.
4. Check that for every α and β, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α.
5. Endow X with a topology induced by $\{\phi_\alpha\}$ i.e. a subset $U \subseteq U_\alpha$ is open in X if and only if $\phi_\alpha(U)$ is open in $\mathbb{C}$.
6. Check that ϕ_α is pairwise compatible.
7. Check that X is connected and Hausdorff.

Example

Let $\mathbb{CP}^1$ denote the complex projective line. Consider the sets

$$U_0 := \{[x_0, x_1] \in \mathbb{CP}^1 \, : \, x \neq 0\}, \quad U_1 := \{[x_0, x_1] \in \mathbb{CP}^1 \, : \, y \neq 0\}.$$

Consider the bijective map $\phi_i : U_i \to \mathbb{C}$ such that $\phi_i([x_j]) \mapsto \frac{x_j}{x_i}$. Clearly $\phi_0 \circ \phi_1^{-1}$ is compatible as it sends s to $\frac{1}{s}$. Give $\mathbb{CP}^1$ the topology induced by the charts.

The complex projective line $\mathbb{CP}^1$ is a Reimann surface.

Functions and Maps

Definition

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Example

Let $f([x, y]) = \frac{p(x,y)}{q(x,y)}$ be a complex-valued function defined on $\mathbb{CP}^1$, where $p(x, y)$ and $q(x, y)$ are homogenous polynomials of the same degree. Suppose $q(x_0, y_0) \neq 0$ then $q \neq 0$ on some neighborhood of $[x_0, y_0]$, implying that f is holomorphic on some neighborhood of $[x_0, y_0]$.

Singularities and Meromorphic Functions

Definition

Let f be a holomorphic function in a punctured neighborhood of $p \in X$.

- We say f has a *removable singularity* at p if and only if there exists a chart $\phi : U \to V$, with $p \in U$, such that the composition $f \circ \phi^{-1}$ has a removable singularity at $\phi(p)$.

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- We say f has a *removable singularity* at p if and only if there exists a chart $\phi : U \to V$, with $p \in U$, such that the composition $f \circ \phi^{-1}$ has a removable singularity at $\phi(p)$.
- We say f has a *pole* at p if and only if there exists a chart $\phi : U \to V$, with $p \in U$, such that the composition $f \circ \phi^{-1}$ has a pole at $\phi(p)$.
- We say f has a *essential singularity* at p if and only if there exists a chart $\phi : U \to V$, with $p \in U$, such that the composition $f \circ \phi^{-1}$ has a essential singularity at $\phi(p)$.

Meromorphic functions

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A function f on X is meromorphic on X if it is either holomorphic, has a removable singularity, or has a pole at every point p in X.

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Example

If $p(x, y)$ and $q(x, y)$ are homogeneous polynomials of the same degree with q not identically zero, then $f(x, y) = \frac{p(x,y)}{q(x,y)}$ descends to a meromorphic function on $\mathbb{CP}^1$.

Definition

1. If W is an open set in a Reimann surface X, we denote $\mathcal{O}(W)$ to be the collection of holomorphic functions on W. We note that $\mathcal{O}(W)$ is a $\mathbb{C}$-algbera.
2. If W is an open set in a Reimann surface X, we denote $\mathcal{M}(W)$ to be the collection of meromorphic functions on W. We note that $\mathcal{M}(W)$ is a $\mathbb{C}$-algbera.

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Definition

Let X and Y be Reimann surfaces. A mapping $F : X \to Y$ is holomorphic if for any point $p \in X$, there exists charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$, such that the composition $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi(p)$.

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Example

Every holomorphic complex-valued function on a Reimann surface is a holomorphic map from X to the complex plane $\mathbb{C}$, considered as a trivial Reimann surface.

Properties of holomorphic maps

Lemma

Let X and Y be Reimann surfaces and $F : X \to Y$ be a holomorphic map between them.

1. Composition of holomorphic maps is holomorphic.
2. The composition of a holomorphic map with a holomorphic function is holomorphic.
3. The composition of a holomorphic map with a meromorphic function is meromorphic.

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Thus for any holomorphic map $F : X \to Y$ between Reimann surfaces, then for any open set $W \subseteq Y$ we have a corresponding $\mathbb{C}$-algebra homomorphism

$$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W)).$$

Definition

An *isomorphism* between Reimann surfaces is a holomorphic map $F : X \to Y$ which is bijective, and whose inverse $F^{-1} : X \to Y$ is holomorphic. A self-isomorphism $F : X \to X$ is called an *automorphism*.

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Theorem

The Reimann Sphere $\mathbb{C}_\infty$ and the projective line $\mathbb{CP}^1$ are isomorphic.

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3. Let X be a compact Reimann surface, and let $F : X \to Y$ be a nonconstant holomorphic map. Then Y is compact and F is onto.

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3. Let X be a compact Reimann surface, and let $F : X \to Y$ be a nonconstant holomorphic map. Then Y is compact and F is onto.
4. Let $F : X \to Y$ be a nonconstant holomorphic map. Then for every $y \in Y$, the preimage of $F^{-1}(y)$ is a discrete subset of X.
5. There is a one-one correspondence between meromorphic functions on X and holomorphic maps $F : X \to \mathbb{C}_\infty$ between X and the Riemann sphere which are not identically ∞, given by $F(x) = f(x)$ if x is not a pole of f and $F(x) = \infty$ if x is a pole of f.

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Order of a meromorphic function

Definition

Let f be a meromorphic function at p, whose Laurent series in local coordinate z is $\sum_n c_n(z - z_0)^n$ where $f(p) = z_0$. The order of f at p, denoted by $\texttt{ord}_p(f)$, is the minimum exponent appearing in the Laurent series i.e.

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Theorem

Suppose f is a meromorphic function at p.

1. Then f is holomorphic at p if and only if $\texttt{ord}_p(f) \geq 0$.
2. Then f has a pole at p if and only if $\texttt{ord}_p(f) < 0$.
3. Then f has neither zero nor a pole at p if and only if $\texttt{ord}_p(f) = 0$.

Let f and g be nonzero meromorphic functions at $p \in X$. Then

1. $\texttt{ord}_p(fg) = \texttt{ord}_p(f) + \texttt{ord}_p(g)$.
2. $\texttt{ord}_p(f/g) = \texttt{ord}_p(f) - \texttt{ord}_p(g)$.
3. $\texttt{ord}_p(1/f) = -\texttt{ord}_p(f)$.
4. $\texttt{ord}_p(f \pm g) \geq \min\{\texttt{ord}_p(f), \texttt{ord}_p(g)\}$.

Multiplicity

Proposition: Local Normal Form

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Proof: Consider charts $\phi_1 : U_1 \to V_1$ on X centered at p (i.e. $\phi_1(p) = 0$) and $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$. Consider the Taylor expansion $\sum_m c_m w^m$ of $\Psi = \phi_2 \circ F \circ \phi_1^{-1}(w)$ on $\phi_1(U_1)$. Then $m \geq 1$ as $T(0) = 0$. Take $T(w) = w^m(S(w))^m$ where $S(w) \neq 0$ on $\phi_1(U_1)$.

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$$\begin{aligned}
\phi_2(F((wS(w) \circ \phi_1)^{-1}(wS(w)))) &= \phi_2(F(\phi_1^{-1} \circ (wS(w))^{-1}(wS(w)))) \\
&= T((wS(w))^{-1}(wS(w))) \\
&= T(w) = (wS(w))^m.
\end{aligned}$$

Multiplicity

Definition

The *multiplicity* of F at p, denoted $\texttt{mult}_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \to z^m$.

If $\texttt{mult}_p(F) \geq 2$, then p is called a *ramification point*.

If p is a ramification point, then $F(p) \in Y$ is a branch point.

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Example

Let X be a Riemann surface. Let $\phi : U \to V$ be a chart map on X. Consider ϕ as a holomorphic map to the trivial Riemann surface $\mathbb{C}$. Then ϕ has multiplicity one at every point of U.

Lemma

Let $F : X \to Y$ be a holomorphic map at $p \in X$. Take any local coordinates z near p and w near $F(p)$ where z_0 and w_0 corresponds to p and $F(p)$ respectively. Then F can be written as $w = h(z)$ where h is a holomorphic function. The multiplicity of F at p is one more than the order of vanishing of the derivative $h'(z_0)$ of h i.e.

$$\mathtt{mult}_p(F) = 1 + \mathtt{ord}_{z_0}\left(\frac{dh}{dz}\right).$$

Degree of a Holomorphic map

Theorem

Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. For each $y \in Y$, define $d_y(F)$ to be the sum of the multiplicities of F at the points of X mapping to y:

$$d_y(F) = \sum_{p \in F^{-1}(y)} \mathtt{mult}_p(F).$$

Then $d_y(F)$ is constant, independent of y.

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Definition

Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces, The degree of F, denoted by $\deg(F)$, is the integer $d_y(F)$ for any $y \in Y$.

Some results

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Proposition

Let f be a nonconstant meromorphic function on a compact Riemann surface X. Then

$$\sum_{p \in X} \mathtt{ord}_p(f) = 0.$$

Euler Number

Definition

Let S be a compact 2-manifold. A *triangulation* of S is a decomposition of S into closed subsets, each homeomorphic to a triangle, such that any two triangles are either disjoint, meet only at a single vertex, or meet only along a single edge.

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Definition

Let S be a compact 2-manifold, possibly with boundary. Suppose a triangulation of S is given, with v vertices, e edges, and t triangles. The *Euler number* of S is the integer $\mathbf{e}(S) = v - e + t$.

Fact!

The Euler number is independent of the choice of triangulations of S.

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Theorem (Riemann-Hurwitz Formula)

Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then

$$\mathbf{e}(X) = \texttt{deg}(F)\mathbf{e}(Y) - \sum_{\text{branch point } p \in X} [1 - \texttt{mult}_p(F)].$$

Proof: Since X is compact, the set of ramification points is finite.

Take a triangulation of Y, such that each branch point of F is a vertex. Assume there are v vertices, e edges, and t triangles. Lift this triangulation to X via the map F, and assume there are v' vertices, e' edges, and t' triangles on X. Note that every ramification point of F is a vertex on X.

Since there are no ramification points over the general point of any triangle, each triangle of Y lifts to deg(F) triangles in X. Thus $t' = \text{deg}(F)t$. Similarly, $e' = \text{deg}(F)e$.

Now, for any $q \in Y$, we have

$$\begin{aligned} |F^{-1}(q)| &= \sum_{p \in F^{-1}(q)} 1 \\ &= \sum_{p \in F^{-1}(q)} 1 + \deg(F) - \deg(F) \\ &= \deg(F) + \sum_{p \in F^{-1}(q)} [1 - \mathtt{mult}_p(F)]. \end{aligned}$$

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Therefore, the total number of preimages of vertices of Y, which is the number v' of vertices of X, is

$$\begin{aligned} v' &= \sum_{\text{branch point } q \text{ of } Y} \left(\deg(F) + \sum_{p \in F^{-1}(q)} [1 - \mathtt{mult}_p(F)] \right) \\ &= \deg(F) v + \sum_{\text{ramification point } p \text{ of } X} [1 - \mathtt{mult}_p(F)] \end{aligned}$$

So,

$$\begin{aligned}\mathbf{e}(X) &= v' - e' + t' \\ &= \left(\deg(F)v + \sum_{\text{vertex } p \text{ of } X} [1 - \texttt{mult}_p(F)]\right) - \deg(F)e + \deg(F)t \\ &= \deg(F)\mathbf{e}(Y) + \sum_{\text{vertex } p \text{ of } X} [1 - \texttt{mult}_p(F)].\end{aligned}$$

□

Example

Let $f(z) = \frac{z^3}{1-z^2}$, considered as a meromorphic function on the Riemann Sphere $\mathbb{C}_\infty$ i.e. $S^2 \xrightarrow{\phi_1} \mathbb{C}_\infty \xrightarrow{f} \mathbb{C}$ where $\phi_1(x, y, w) = \frac{x}{1-w} + \frac{y}{1-w}\iota = z$ and $\phi_1(0, 0, 1) = \infty$.

At $z = 0$, we have $\frac{z^3}{1-z^2} = \sum_{n\geq 0} z^{2n+3}$, which implies $\mathtt{ord}_0(f) = 3$.

At $z = 1$, we have $\frac{z^3}{1-z^2} = \frac{(1-(1-z))^3}{(1-z)(1+z)}$, which implies $\mathtt{ord}_1(f) = -1$.

At $z = -1$, we have $\frac{z^3}{1-z^2} = \frac{((1+z)-1)^3}{(1-z)(1+z)}$, which implies $\mathtt{ord}_{-1}(f) = -1$.

At $z = \infty$, we have $\frac{\frac{1}{z^3}}{1-\frac{1}{z^2}} = -\sum_{n\geq 0} z^{2n-1}$, which implies $\mathtt{ord}_\infty(f) = -1$.

To find multiplicity, we need to resort to the local coordinates. We have two charts U_1 and U_2 on the domain space along with V_1 and V_2 in the target space. Local coordinates attached to them are $z, \frac{1}{z}, f(z)$ and $\frac{1}{f(z)}$ respectively. We then have

$$h(z) = \begin{cases} f(z) = \frac{z^3}{1-z^2}, & F^{-1}(V_1) \cap U_1, \\ f(\frac{1}{z}) = \frac{1}{z(z^2-1)}, & F^{-1}(V_1) \cap U_2, \\ \frac{1}{f(z)} = \frac{1-z^2}{z^3}, & F^{-1}(V_2) \cap U_1, \\ \frac{1}{f(\frac{1}{z})} = z(z^2-1), & F^{-1}(V_2) \cap U_2. \end{cases}$$

On $F^{-1}(V_1) \cap U_1$, we have $h'(z) = \frac{z^2(3-z^2)}{(1-z^2)^2}$. This gives the ramification points

$$\begin{aligned} z &= 0, \quad \mathtt{mult}_0(F) = 1 + 2 = 3, \\ z &= \sqrt{3}, \quad \mathtt{mult}_{\sqrt{3}}(F) = 1 + 1 = 2, \\ z &= -\sqrt{3}, \quad \mathtt{mult}_{-\sqrt{3}}(F) = 1 + 1 = 2. \end{aligned}$$

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Similarly, we have

$$F^{-1}(V_1) \cap U_2 \implies h'(z) = \frac{1 - 3z^2}{z^2(z^2 - 1)} \qquad \text{R.P } \pm\frac{1}{\sqrt{3}} \sim \pm\sqrt{3}$$

$$F^{-1}(V_2) \cap U_1 \implies h'(z) = \frac{z^2(z^2 - 3)}{z^6} \qquad \text{No new R.P}$$

$$F^{-1}(V_2) \cap U_2 \implies h'(z) = 3z^2 - 1 \qquad \text{No new R.P}$$

Recall the formula

$$\mathbf{e}(X) = \texttt{deg}(F)\mathbf{e}(Y) - \sum_{\text{branch point } p \in X} [1 - \texttt{mult}_p(F)].$$

We know that $\mathbf{e}(\mathbb{C}_\infty) = -2$. Furthermore, the degree of F is 3 by the ramification point 0. So,

$$-2 = 3(-2) - ((1-3) + (1-2) + (1-2))$$

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THANK YOU!!!