Classifying Irreducible Unipotent Numerical Monoids into Symmetric and Pseudo Symmetric Monoids

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Numerical Semigroups

We assume $\mathbb{N} = \{0, 1, 2, 3, \rightarrow\}$ throughout the talk.

Definition

A subset $\mathcal{S} \subseteq \mathbb{N}$ is a numerical semigroup if

- $0 \in \mathcal{S}$.
- If $a, b \in \mathcal{S}$ then $a + b \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbb{N}$ is finite.

pause

Example

Let $\mathcal{S} = \{0, 3, 5, 6, 8, 9, 10, \rightarrow\} = \langle 3, 5 \rangle$.

Invariants

Let $\mathcal{S}$ be a numerical semigroup.

- Multiplicity $\mathtt{m}(\mathcal{S})$ is the smallest non-zero number in $\mathcal{S}$.
- Gap set $\mathtt{G}(\mathcal{S})$ is the set of elements of the complement of $\mathcal{S}$ in $\mathbb{Z}_{\geq 0}$. Genus $\mathtt{g}(\mathcal{S})$ is the cardinality of $\mathtt{G}(\mathcal{S})$.
  pause
- Frobenius element $\mathtt{F}(\mathcal{S})$ is the largest number in the gap set $\mathtt{N}(\mathcal{S})$.
- Conductor $\mathtt{c}(\mathcal{S}) = \mathtt{F}(\mathcal{S}) + 1$.
- The Pseudo-Frobenius set is defined as
  $\mathtt{PF}(\mathcal{S}) := \{x \in \mathtt{G}(\mathcal{S}) : x + \mathcal{S} \subseteq \mathcal{S}\}$.
  pause
- Sporadic elements $\mathtt{N}(\mathcal{S}) := \{x \in \mathcal{S} : x < \mathtt{F}(\mathcal{S})\}$. We denote $\mathtt{n}(\mathcal{S}) = |\mathtt{N}(\mathcal{S})|$.
- Minimal generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$.

Example

Let $\mathcal{S} = \{0, 3, 6, 8, 9, 10, \rightarrow\} = \langle 3, 8, 10 \rangle$.

- $\mathtt{m}(\mathcal{S}) = 3$.
- $\mathtt{G}(\mathcal{S}) = \{1, 2, 4, 5, 7\}$ and $\mathtt{g}(\mathcal{S}) = 5$.
  pause
- $\mathtt{F}(\mathcal{S}) = 7$.
- $\mathtt{c}(\mathcal{S}) = 8$.
- $\mathtt{PF}(\mathcal{S}) = \{5, 7\}$
  pause
- $\mathtt{N}(\mathcal{S}) = \{0, 3, 6\}$ and $\mathtt{n}(\mathcal{S}) = 3$.
- $\mathtt{e}(\mathcal{S}) = 3$

Definition

A numerical semigroup $\mathcal{S}$ is said to be irreducible if it cannot be expressed as the intersection of two distinct numerical semigroups properly containing $\mathcal{S}$.

pause

Example

The numerical semigroup $\mathcal{S} = \langle 3, 7, 11 \rangle$ is irreducible.

pause

$$
\begin{tikzcd}[arrows=dash]
 & \mathbb{N} \arrow[d] & \\
 & \langle 2, 3 \rangle \arrow[d] & \\
 & \langle 3, 4, 5 \rangle \arrow[dl] \arrow[dr] & \\
\langle 3, 4 \rangle \arrow[dr] & & \langle 3, 5, 7 \rangle \arrow[dl] \\
 & \langle 3, 7, 8 \rangle &
\end{tikzcd}
$$

Definition

Let $\mathcal{S}$ be a numerical semigroup. Let $\mathrm{g}(\mathcal{S})$ denote the genus of $\mathcal{S}$.

- We say that S is *symmetric* if $\mathrm{g}(\mathcal{S}) = \frac{1+\mathtt{F}(\mathcal{S})}{2}$.
- We say that S is *pseudo-symmetric* if $\mathrm{g}(\mathcal{S}) = \frac{2+\mathtt{F}(\mathcal{S})}{2}$.

pause

Example

$$\mathcal{S} = \langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, \rightarrow\}.$$

pause We have $\mathtt{G}(\mathcal{S}) = \{1, 2, 4, 7\}$ so, $\mathrm{g}(\mathcal{S}) = 4$. Note $\mathtt{F}(\mathcal{S}) = 7$.
pause Thus, $\mathcal{S}$ is Symmetric.

pause

Example

$$\mathcal{S} = \langle 3, 7, 11 \rangle = \{0, 3, 6, 7, 9, 10, 11, \rightarrow\}.$$

pause We have $\mathtt{G}(\mathcal{S}) = \{1, 2, 4, 5, 8\}$ so, $\mathrm{g}(\mathcal{S}) = 5$. Note $\mathtt{F}(\mathcal{S}) = 8$.
pause Thus, $\mathcal{S}$ is Pseudo-symmetric.

Theorem

Let $\mathcal{S}$ be an irreducible numerical semigroup. Then $\mathcal{S}$ is either symmetric or pseudo-symmetric. Moreover, every symmetric or pseudo-symmetric semigroup are irreducible.

Applications

- Let $\mathcal{S}$ be a numerical semigroup. Let $\mathbb{K}$ be algebraically closed and define $\mathbb{K}[\mathcal{S}] = \oplus_{s \in \mathcal{S}} \mathbb{K} t^s$. Consider the ring $\mathbb{K}[[\mathcal{S}]]$.
  [Kun70] showed that $\mathbb{K}[[\mathcal{S}]]$ is a Gorenstein ring if and only if $\mathcal{S}$ is symmetric.
  pause *Remark:* A Noetherian ring R is Gorenstein if R has finite injective dimension as an R-module.

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Unipotent Numerical Semigroups

Let

$$\mathbf{U}(n,\mathbb{N}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{ij}\}_{1<i<j<n} \in \mathbb{N} \right\}$$

pause We fix a finitely generated monoid $\mathbf{G} \subseteq \mathbf{U}(n,\mathbb{N})$.
pause A subset $\mathcal{S} \subseteq \mathbf{G}$ is a *unipotent numerical semigroup* if

- $\mathbf{1}_n \in \mathcal{S}$.
- If $A, B \in \mathcal{S}$ then $AB \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbf{G}$ is finite.

Let us fix $\mathbf{G} = \mathbf{P}(n, \mathbb{N})$ where

$$\mathbf{P}(n, \mathbb{N}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{1j}\}_{1<j\leq n} \in \mathbb{N} \right\} \quad (\cong \mathbb{N}^{n-1})$$

Example (k-th Fundamental monoid)

Let

$$\mathbf{P}_k(n) := \{(x_{1j})_{1<j\leq n} \in \mathbf{P}(n) : x_{1j} \geq k \text{ for some } 1 > j \leq n\} \subseteq \mathbf{P}(n).$$

pause

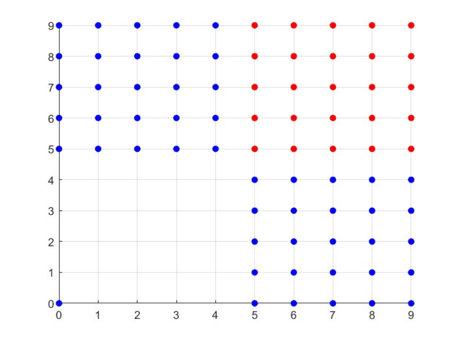


Figure: This is $\mathbf{P}_5(3)$

Example

Let $\mathcal{S} \subseteq \mathbf{P}(3)$ and consider $\mathcal{S}$ plotted as

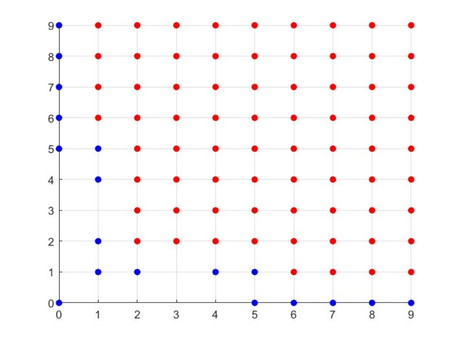


Figure: $\mathcal{S} = \langle (1,1), (2,1), (1,2), (4,1), (1,4), \mathbf{P}_5 \rangle$

Notation

1. Let $\mathbf{G_U}$ denote $\mathbf{U}(n, \mathbb{N})$ and $\mathbf{G_P}$ denote $\mathbf{P}(n, \mathbb{N})$. Collectively we denote them by $\mathbf{G}$.
2. We let $\mathbf{G}_k$ denote the corresponding k-th Fundamental monoid in $\mathbf{G_U}$ or $\mathbf{G_P}$.
   pause
3. An asterisk on a set denote the set minus the identity element e.g. $\mathbf{G}^* = \mathbf{G} \backslash \mathbf{1}_n$ where $\mathbf{1}_n$ denote the $n \times n$ identity matrix.
4. We will denote an arbitrary Unipotent Numerical Monoid in $\mathbf{G}$ by $\mathcal{S}$.

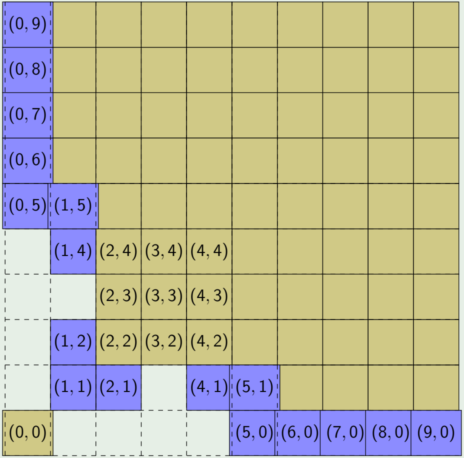
Invariants

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$.

- Gap set $\mathtt{G}(\mathcal{S})$ is the set of elements of the complement of $\mathcal{S}$ in $\mathbf{G}$. Genus $\mathtt{g}(\mathcal{S}) = |\mathtt{G}(\mathcal{S})|$.
- Generating number $\mathtt{r}(\mathcal{S}) = \min\{k \in \mathbb{N} : \mathbf{G}_k \subseteq \mathcal{S}\}$.
  pause
- Sporadic elements $\mathtt{N}(\mathcal{S}) := \mathcal{S} \backslash \mathbf{G}_{\mathtt{r}(\mathcal{S})}$ and $\mathtt{n}(\mathcal{S}) = |\mathtt{N}(\mathcal{S})|$.
- Minimal generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$.

Example

Let $\mathcal{S} = \langle (1,1),(1,2),(1,4),(2,1),(4,1)\rangle \sqcup \mathbf{G}_5$ in $\mathbf{G} = \mathbf{G_P} = \mathbf{P}(3,\mathbb{N})$.



$\mathtt{r}(\mathcal{S}) = 5, \quad \mathtt{g}(\mathcal{S}) = 10, \quad \mathtt{e}(\mathcal{S}) = 17, \quad \mathtt{n}(\mathcal{S}) = 15$

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. The different types of *Frobenius sets* in $\mathcal{S}$ are defined as follows:

- The *left Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{F}_l(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } A\mathbf{G}^* \subseteq \mathcal{S}\}.$$

  pause

- The *right Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{F}_r(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } \mathbf{G}^*A \subseteq \mathcal{S}\}.$$

- The *two-sided Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{F}_t(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } \{A\mathbf{G}^*, \mathbf{G}^*A\} \subseteq \mathcal{S}\}.$$

Remark: If $\mathbf{G} = \mathbf{G_P}$, then all the three types of Frobenius sets coincides.

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. The different types of *Pseudo-Frobenius sets* in $\mathcal{S}$ are defined as follows:

- The *left Pseudo-Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{PF}_l(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } A\mathcal{S}^* \subseteq \mathcal{S}\}.$$

  pause

- The *right Pseudo-Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{PF}_r(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } \mathcal{S}^*A \subseteq \mathcal{S}\}.$$

- The *two-sided Pseudo-Frobenius set* of $\mathcal{S}$ is defined as

$$\mathtt{PF}_t(\mathcal{S}) := \{A \in \mathbf{G} : A \notin \mathcal{S} \text{ and } \{A\mathcal{S}^*, \mathcal{S}^*A\} \subseteq \mathcal{S}\}.$$

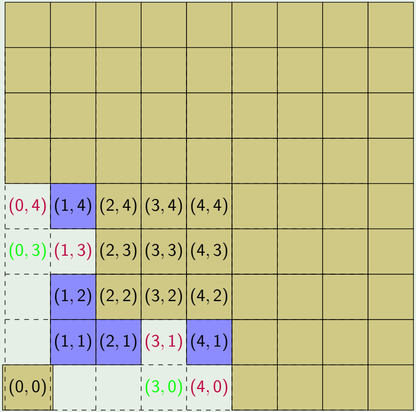
Remark: If $\mathbf{G} = \mathbf{G_P}$, then all the three types of Pseudo-Frobenius sets coincides.

Definition

A unipotent numerical monoid $\mathcal{S}$ in $\mathbf{G}$ is said to be irreducible if it cannot be expressed as the intersection of two distinct unipotent numerical monoids properly containing $\mathcal{S}$.

Example (Not Irreducible)

Let $\mathcal{S} = \langle (1,1), (1,2), (1,4), (2,1), (4,1) \rangle \sqcup \mathbf{G}_5$ in $\mathbf{G} = \mathbf{G}_\mathbf{P} = \mathbf{P}(3, \mathbb{N})$.

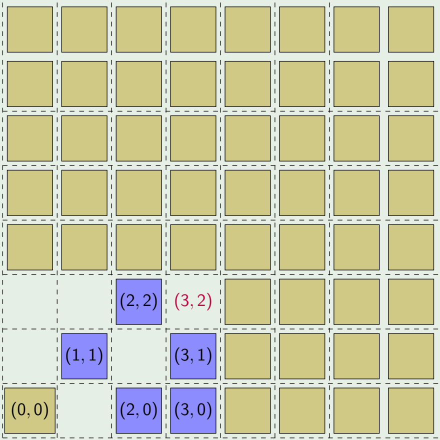


pause

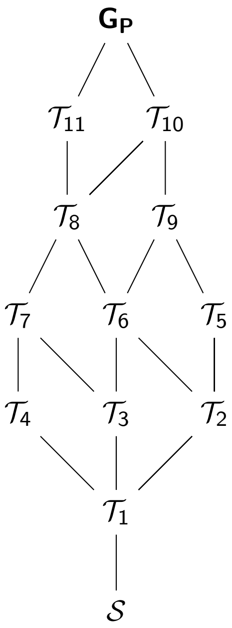
$\mathtt{F}(\mathcal{S}) = \{(0,4), (1,3), (3,1), (4,0)\}, \quad \mathtt{PF}(\mathcal{S}) = \mathtt{F}(\mathcal{S}) \cup \{(0,3), (3,0)\}$

Example (Irreducible)

Let $\mathcal{S} = \langle (1,1),(2,0),(2,2),(3,0),(3,1)\rangle \sqcup \mathbf{G}_4$ in $\mathbf{G} = \mathbf{G_P} = \mathbf{P}(3,\mathbb{N})$.



$\mathrm{F}(\mathcal{S}) = \{(3,2)\}, \quad \mathrm{PF}(\mathcal{S}) = \mathrm{F}(\mathcal{S})$


$\mathbf{G_P}$
$\mathcal{T}_{11}$
$\mathcal{T}_{10}$
$\mathcal{T}_{8}$
$\mathcal{T}_{9}$
$\mathcal{T}_{7}$
$\mathcal{T}_{6}$
$\mathcal{T}_{5}$
$\mathcal{T}_{4}$
$\mathcal{T}_{3}$
$\mathcal{T}_{2}$
$\mathcal{T}_{1}$
$\mathcal{S}$

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Definition

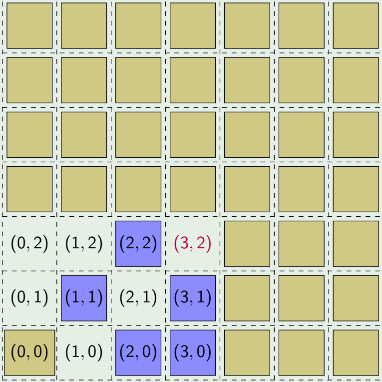
Let $\mathcal{S}$ be an irreducible unipotent numerical monoid in $\mathbf{G}$.

- Then $\mathcal{S}$ is called *symmetric* if for every $A \in \mathbf{G} \backslash \mathcal{S}$, we have $\mathtt{F}_t(\mathcal{S}) \cap (A\mathcal{S} \cap \mathcal{S}A) \neq \emptyset$.
  pause
- Then $\mathcal{S}$ is called *pseudo-symmetric* if for every $A \in \mathbf{G} \backslash \mathcal{S}$, we have at least one of the following 2 cases:
  1. We have $A^2 \in \mathtt{F}_t(\mathcal{S})$.
  2. We have $\mathtt{F}_t(\mathcal{S}) \cap (A\mathcal{S} \cap \mathcal{S}A) \neq \emptyset$.

Symmetric Example

Example

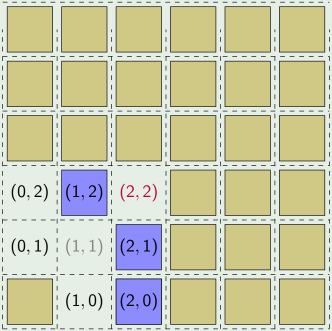
Let $\mathcal{S} = \langle (1,1),(2,0),(2,2),(3,0),(3,1)\rangle \sqcup \mathbf{G}_4$ in $\mathbf{G} = \mathbf{G}_\mathbf{P} = \mathbf{P}(3,\mathbb{N})$.



Pseudo-Symmetric Example

Example

Let $\mathcal{S} = \langle (1,2), (2,0), (2,1) \rangle \sqcup \mathbf{G}_3$ in $\mathbf{G} = \mathbf{G_P} = \mathbf{P}(3, \mathbb{N})$.



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Observation

Let $\mathcal{S}$ be a unipotent numerical monoid in **G**. If $\mathcal{S}$ is irreducible then $|\mathtt{F}_l(\mathcal{S})| = |\mathtt{F}_r(\mathcal{S})| = |\mathtt{F}_t(\mathcal{S})| = 1$

pause

Lemma (Can, S. 23)

Let $\mathcal{S}$ be a unipotent numerical monoid in **G**. If $|\mathtt{F}(\mathcal{S})| = 1$ for any one type of Frobenius set, then the following statements are equivalent.

- $\mathcal{S}$ is irreducible.
  pause
- $\mathcal{S}$ is maximal with respect to set inclusion in the set of unipotent numerical monoid $\mathcal{S}$ such that $\mathtt{F}(\mathcal{S}) \cap \mathcal{T} = \emptyset$.

pause Furthermore, if $|\mathtt{F}_l(\mathcal{S})| = |\mathtt{F}_r(\mathcal{S})| = |\mathtt{F}_t(\mathcal{S})| = 1$, then we can add the following equivalent statement to the above list.

- $\mathcal{S}$ is maximal with respect to set inclusion in the set of unipotent numerical monoid $\mathcal{S}$ such that $\mathtt{F}(\mathcal{S}) = \mathtt{F}(\mathcal{T})$.

Theorem (Can, S. 23)

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. If one of the following conditions holds true, then $\mathcal{S}$ is irreducible.

- If $|\mathtt{F}_l(\mathcal{S})| = 1$ and for every $A \in \mathbf{G} \setminus \mathcal{S}$, we have $\mathtt{F}_l(\mathcal{S}) \cap A\mathcal{S} \neq \emptyset$.
- If $|\mathtt{F}_r(\mathcal{S})| = 1$ and for every $A \in \mathbf{G} \setminus \mathcal{S}$, we have $\mathtt{F}_l(\mathcal{S}) \cap \mathcal{S}A \neq \emptyset$.
- If $|\mathtt{F}_t(\mathcal{S})| = 1$ and for every $A \in \mathbf{G} \setminus \mathcal{S}$, we have $\mathtt{F}_l(\mathcal{S}) \cap (A\mathcal{S} \cap \mathcal{S}A) \neq \emptyset$.

pause *Remark:* This shows that the condition of irreducibility in symmetricity can be dropped and be replaced by $|\mathtt{F}(\mathcal{S})| = 1$.

Main Result

Theorem (Can, S. 23)

Let $\mathcal{S}$ be a unipotent numerical monoid. If $\mathcal{S}$ is irreducible, then $\mathcal{S}$ is either symmetric or pseudo-symmetric.

Note that for symmetric numerical semigroups, we had the definition to be $\mathrm{g}(\mathcal{S}) = \frac{1+\mathrm{F}(\mathcal{S})}{2}$. We have an equivalent description for unipotent numerical monoid.

pause We introduce the partial the following partial order on $\mathbf{G}$. Let $A, B \in \mathbf{G}$, then

$$A \leq_{\mathbf{G},t} B \iff \{BA^{-1}, A^{-1}B\} \subseteq \mathbf{G}.$$

pause For example, if $\mathbf{G} = \mathbf{G}_{\mathbf{P}} = \mathbf{P}(3, \mathbb{N})$ then

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leq_{\mathbf{G},t} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ as } \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proposition (Can, S. 23)

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$ such that $|\mathtt{F}_t(\mathcal{S})| = 1$. Then we have

$$|\mathtt{n}(\mathcal{S}, \mathtt{F}, \preceq)| = \mathtt{g}(\mathcal{S})$$

where

$$\mathtt{n}(\mathcal{S}, \mathtt{F}, \preceq) := \{B \in \mathcal{S} : \mathbf{1}_n \leq_{\mathbf{G},t} B \leq_{\mathbf{G},t} \mathtt{F}\}$$

Future directions & problems

1. Connection to commutative algebra.
2. Characterize irreducibility with respect to the set of divisors

$$\mathbf{D}(X) := \{A \in \mathcal{S} : A \leq_{\mathcal{S},t} X\}.$$

3. Derive connection to Weierstrass semigroup of multiple points on a curve X.
4. Connection to algebraic coding theory.
   pause
5. We look forward to generalizing it to linear algebraic groups. We know that $\mathbf{G} = \mathbf{R} \ltimes \mathbf{U}(n)$.

References

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THANK YOU!!!