Fibonacci Sequence, Golden Ratio, and Sunflowers

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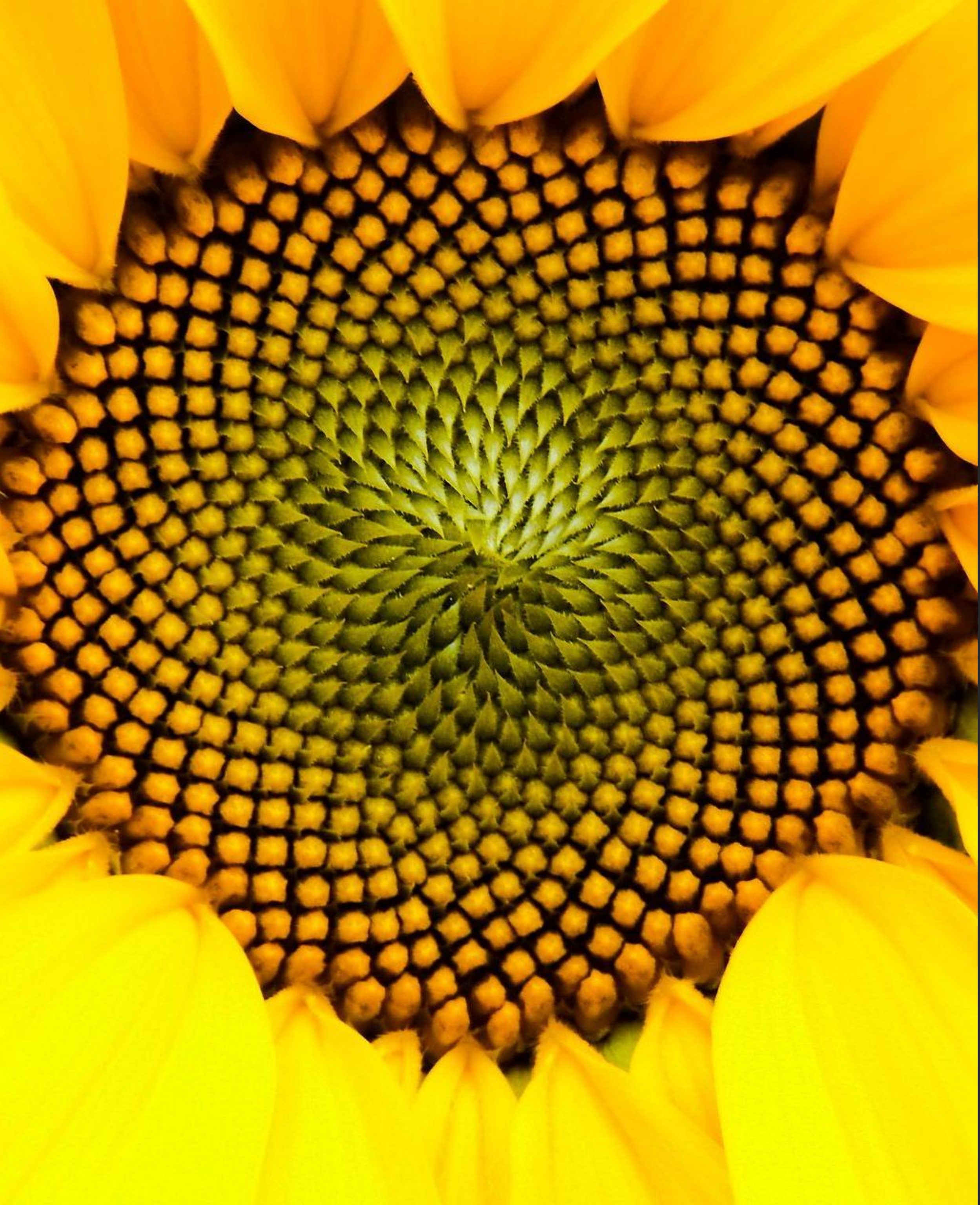

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Fibonacci Sequence

Leonardo of Pisa (Fibonacci) (1180-1250)

- The problem from which the sequence arose was presented by Fibonacci. The problem was

"A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive? "

Growth of rabbit colony

Months	Adult pairs	Young pairs	Total
1	1	1	2
2	2	1	3
3	3	2	5
4	5	3	8
5	8	5	13
6	13	8	21
7	21	13	34
8	34	21	55
9	55	34	89
10	89	55	144
11	144	89	233
12	233	144	377

- **We define Fibonacci sequence as**

$$F_0 = 1, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}$$

- **The first few terms of the sequence are**

$$1\ \ 1\ \ 2\ \ 3\ \ 5\ \ 8\ \ 13\ \ 21\ \ 34\ \ 55\ \ 89\ \ 144\ \ 233\ \ 377\ \ \cdots$$

- **This sequence has many interesting properties**
- $\gcd(F_n, F_{n+1}) = 1\ \forall\, n \in \mathbb{N}$.

- For $m \geq 1, \mathrm{n} \geq 1$, F_{mn} **is divisible by** F_m.

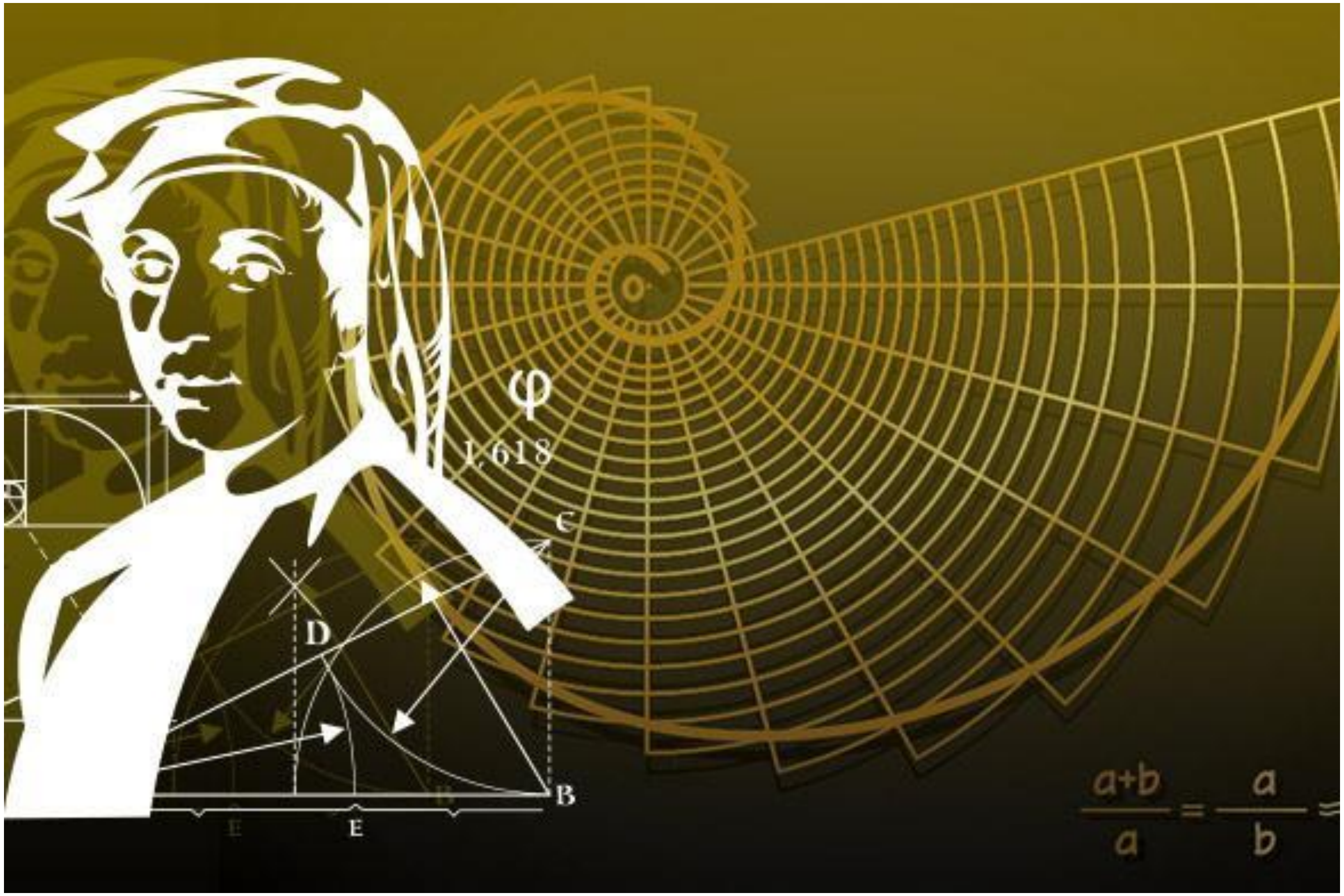
- **The gcd of two Fibonacci numbers is again a Fibonacci number; more specifically**

$$\gcd(F_m, F_n) = F_d, \qquad where\ \ d = \gcd(m, n)$$

- **Finally**

$$F_m | F_n \quad if\ and\ only\ if \quad m|n\ \ for\ n \geq m \geq 3.$$

The Golden Ratio



- The history of Golden Ratio dates back to the era of the Greeks Mathematicians (Pythagoras etc.). The Greeks says that a rectangle of sides a and b where $a > b$, satisfying

$$\frac{a+b}{a} = \frac{a}{b}$$

is the most pleasing to the eye.

- In other words, two quantities are in the Golden Ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities.
- If we take $a = x$ and $b = 1$, we get

$$\frac{1+x}{x} = x \quad \to \quad x^2 - x - 1 = 0$$

Solving it, we get

$$x = \frac{1 \pm \sqrt{5}}{2} \quad \to \quad x = 1.618033 \ldots \quad or \quad x = -0.618033$$

$$Golden\ Ratio = \phi = 1.618033\ldots$$

- **Interestingly, Interestingly, It so happens that the ratio of consecutive Fibonacci Sequence converges to the Gold Ratio i.e.**

$$1 \;\; 1 \;\; 2 \;\; 3 \;\; 5 \;\; 8 \;\; 13 \;\; 21 \;\; 34 \;\; 55 \;\; 89 \;\; 144 \;\; 233 \;\; 377 \;\; \cdots$$

$$\frac{3}{2} = 1.5$$

$$\frac{5}{3} = 1.6667,$$

$$\vdots$$

$$\frac{144}{89} = 1.6179$$

$$\frac{233}{144} = 1.6181$$

$$\frac{377}{233} = 1.6180$$

$$\vdots$$

- Just before going any further let's recall the definition of rational and irrational numbers.
- Those numbers which can be written in the form of $\frac{a}{b}$ where $a, b \neq 0 \in \mathbb{Z}$ are called rational numbers. Numbers which are not rational are called irrational numbers.

Continued Fractions

- This is a very interesting way in number theory to understand rational and irrational numbers.

$$5.56 = 5 + \frac{56}{100} = 5 + \frac{1}{\frac{100}{56}} = 5 + \frac{1}{1 + \frac{11}{14}} = 5 + \frac{1}{1 + \frac{1}{1 + \frac{3}{11}}} = \cdots$$

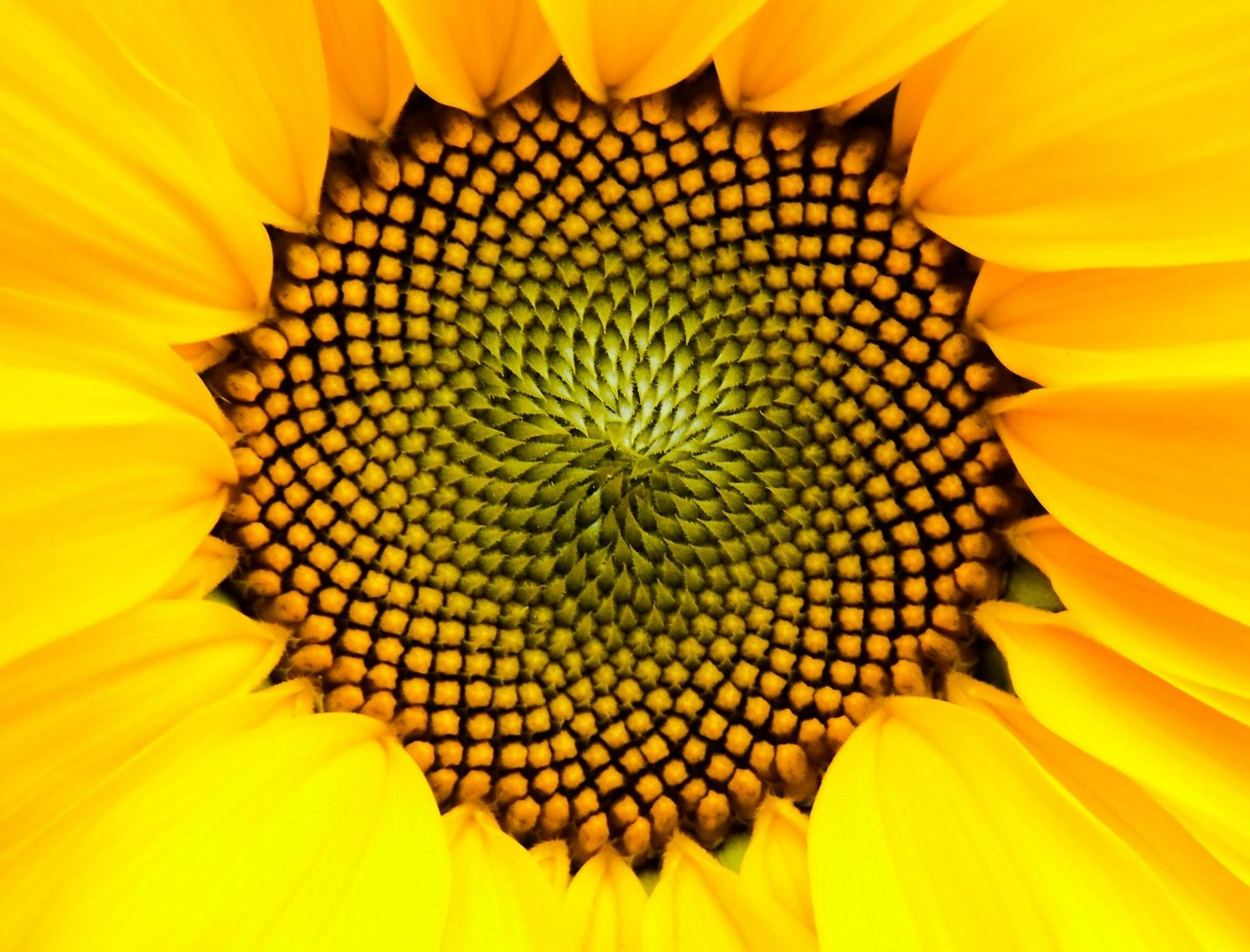
$$5.56 = 5 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{2}}}}}$$

- Any rational number can be written as a finite simple continued fraction.
- Any irrational number cannot be written as a finite simple continued fraction.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cdots}}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}}$$

Why all of this is important and why haven't we seen Sunflowers yet ????



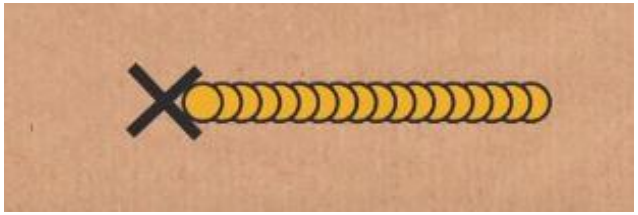
- The pattern of seed on a growing sunflowers obey a law.
- For growing sunflower, it is beneficial to push out each new floret is as far as possible from the existing florets. That gives each floret the most spaces to grow.
- Studies have shown that under many growth conditions, each floret should emerge at an angle $137.5°$. This is also known as the Golden angle as

$$137.5° \approx 360° - \frac{360°}{\phi}$$

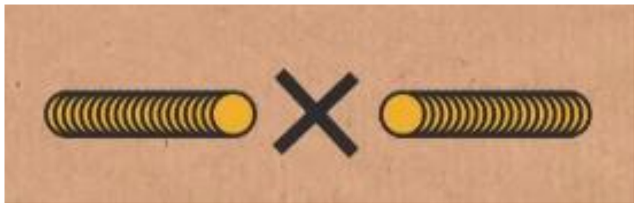
- The relationship between the Golden Angle, the Golden Ration and the Fibonacci Sequence is what causes the sunflowers spirals to appear out of numbers straight out of Fibonacci Sequence.

- Suppose we are the Sunflower, and we want to place our florets in such a way that each new floret is as far as possible from the existing florets. Let r denote the fraction degree of rotation.

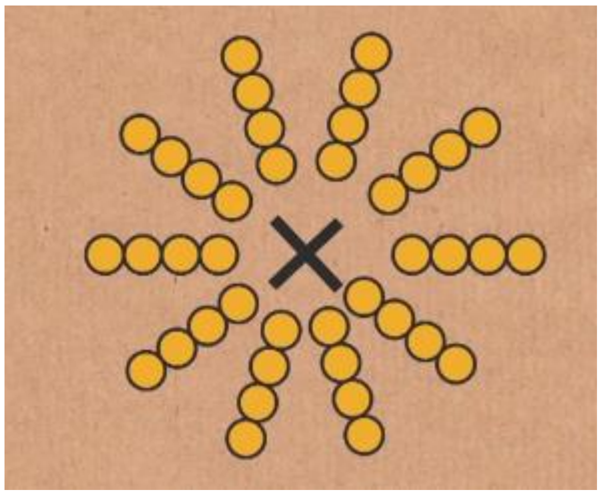
$$r = 0$$



$$r = \frac{1}{2}$$



$$r = \frac{1}{10}$$

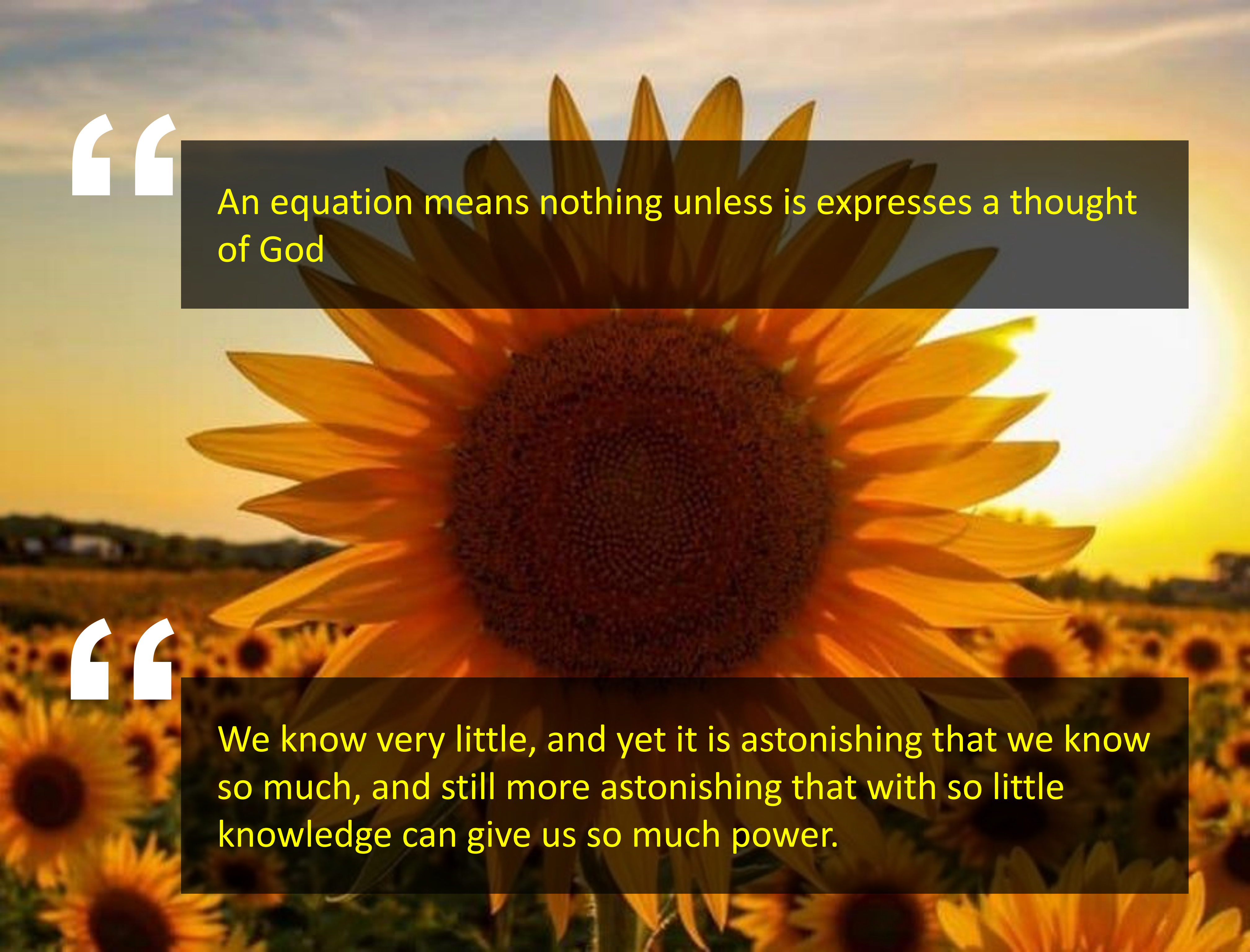


- Recall our continued fraction of π.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cdots}}}}} = \frac{355}{113}$$

- This says that π is not that much irrational in the sense that it can be easily approximated by a rational number.
- In the sense, the most irrational number would be which could hardly be approximated by any rational number. So, the number should be of the type

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \quad \rightarrow \quad x = 1 + \frac{1}{x} \quad \rightarrow \quad x^2 - x - 1 = 0$$


“
An equation means nothing unless is expresses a thought of God
“
We know very little, and yet it is astonishing that we know so much, and still more astonishing that with so little knowledge can give us so much power.

References

- Number Theory by David M. Burton
- The Golden Ratio and Fibonacci Numbers by Richard A. Dunlap
- Numberphile – Youtube


Thank You

18 March 2021