The 7 Millennium Prize Problems

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Clay Mathematical Institute

In the year 2000, the Clay Mathematical Institute presented these 7 problems as a challenge to all the Mathematicians of the 21st century. It is hoped that the problems will be solved in this century. To put in some encouragement, beside gaining immortality in Mathematics, a 1-million-dollar prize will also be awarded to the problem solver.

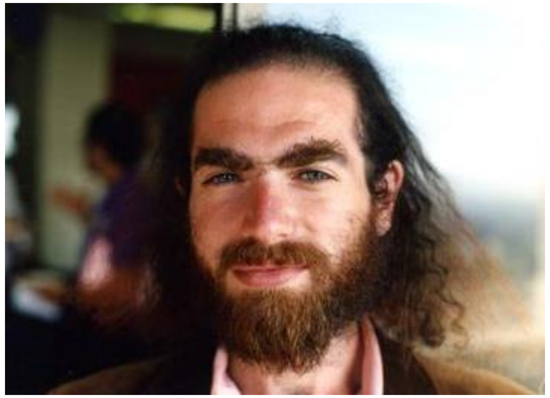
The Problems

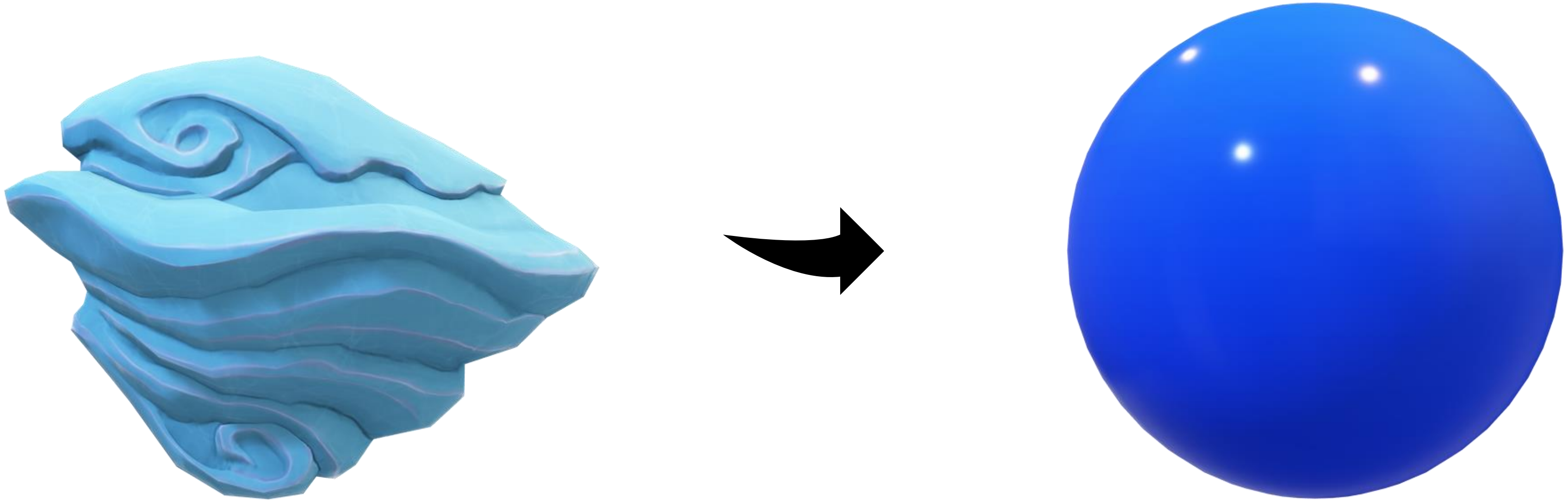
- The Poincare Conjecture (Solved by Grigori Perelman in 2003)
- The Birch and Swinnerton-Dyer Conjecture
- The Reimann Hypothesis
- Existence and Smoothness of the Navier-Stokes Equation
- The Hodge Conjecture
- The P versus NP Problem
- Quantum Yang-Mills Theory

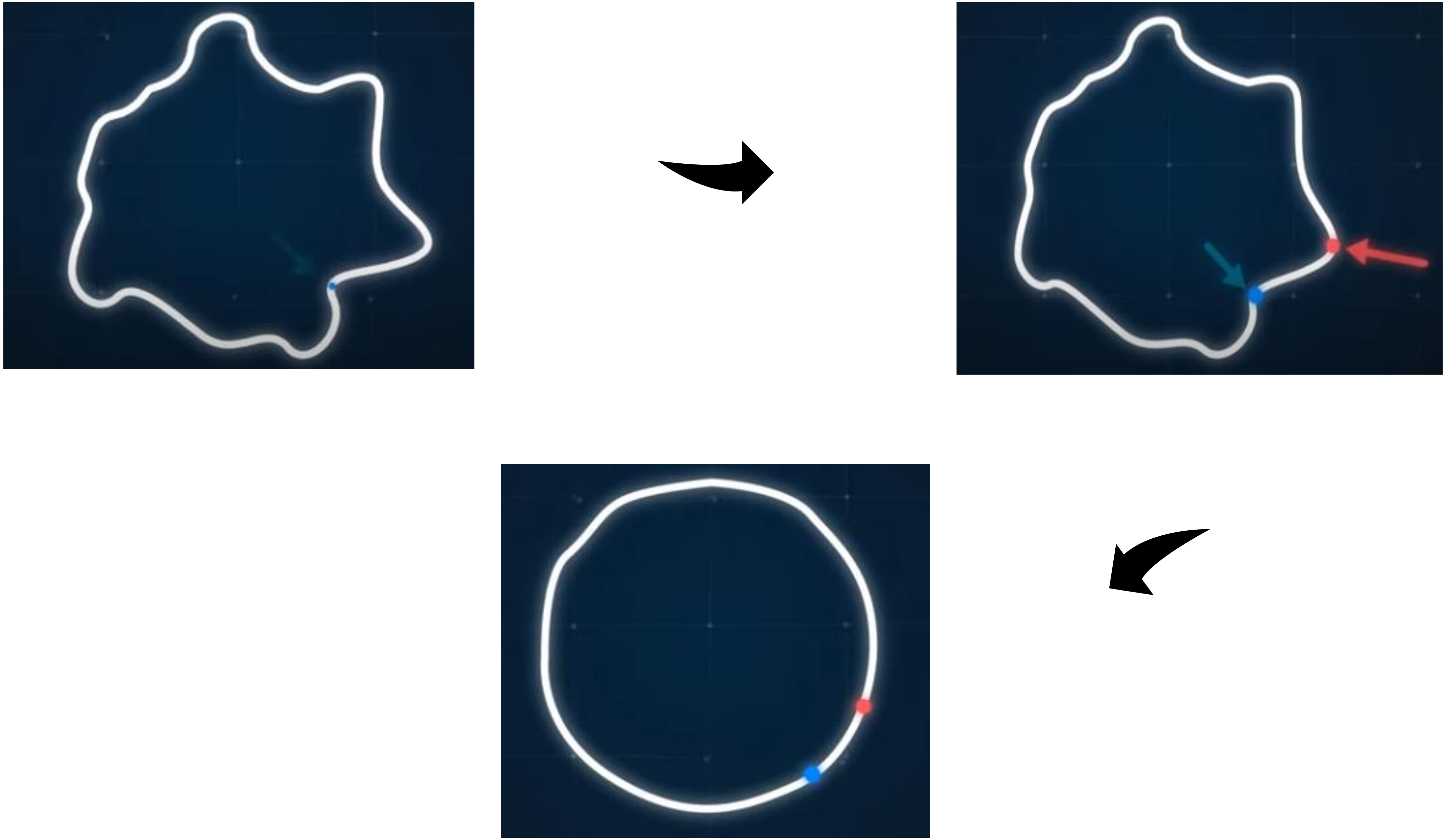
1. The Poincare Conjecture

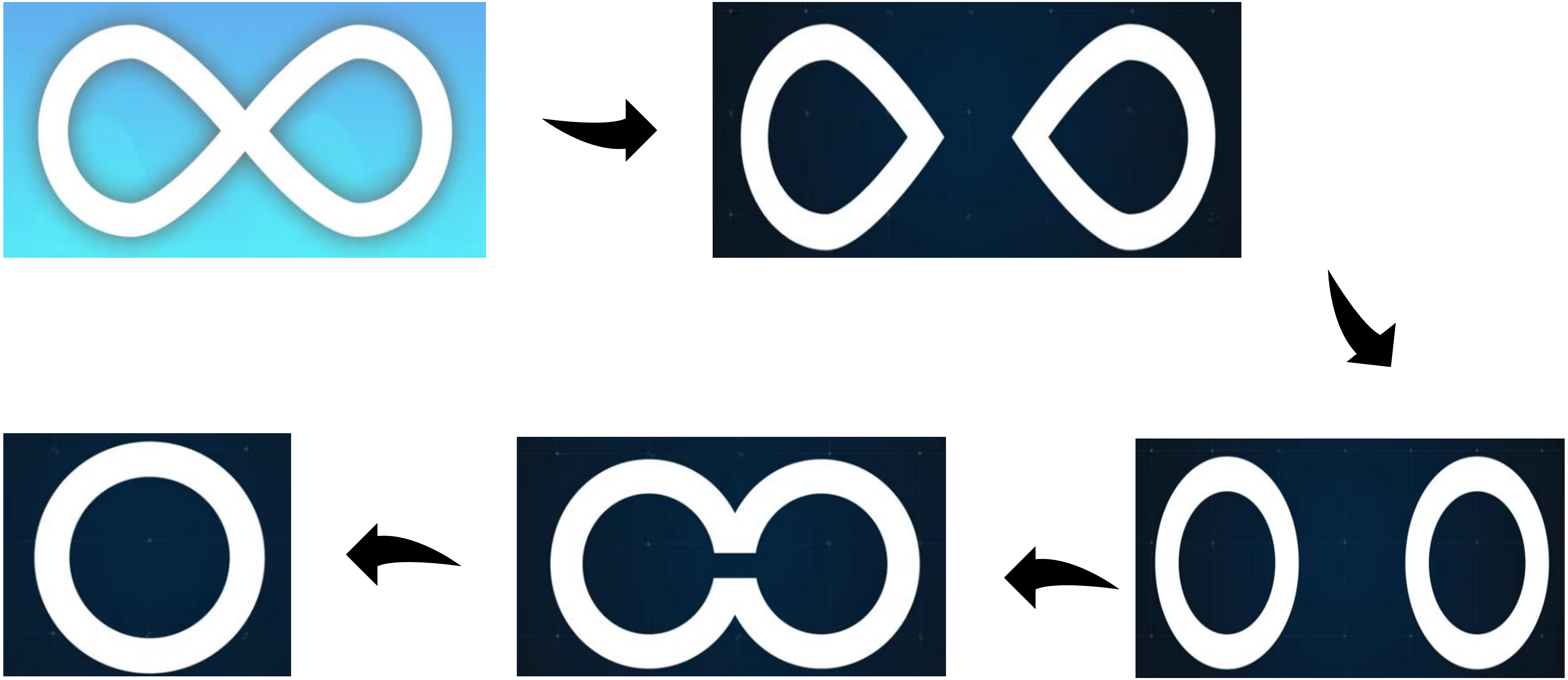
An overview to the conjecture.

- The problem was presented by Henri Poincare in 1904.
- The problem is of the field Topology.
- The conjecture roughly states that any 3-dimensional figure which does not have any holes in it, can be molded or reshaped (by stretching and squeezing, not tearing apart) to a 3-D sphere.
- Mathematically it is formulated as every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.
- Interestingly the conjecture is true for any dimension. It was proved for higher dimensions but was hard to prove for the 3 dimension.
- Grigori Perelman extended the idea of Ricci flow and used it to solve the conjecture.





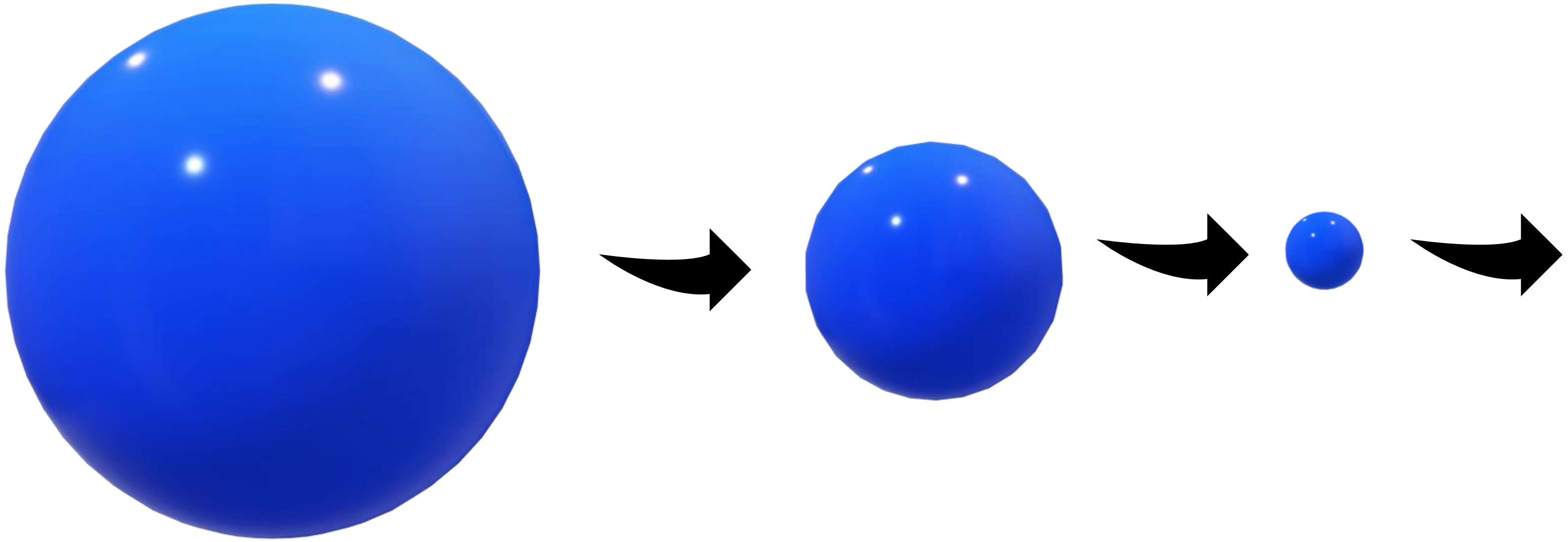




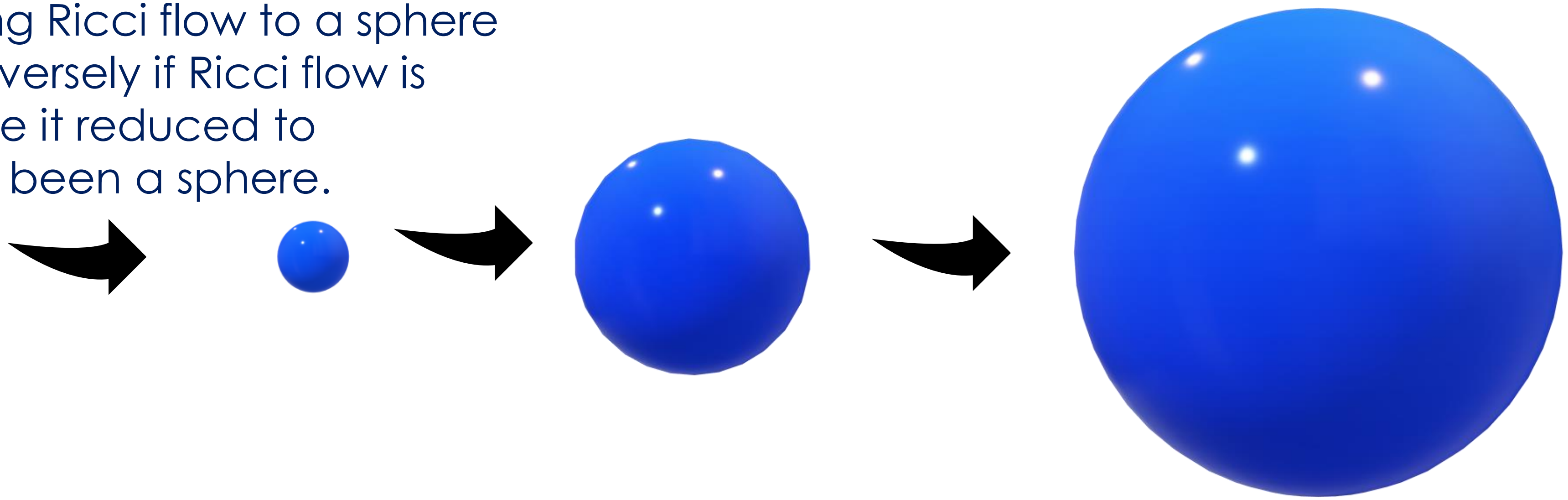
$$2R = -\frac{d}{dt}g$$

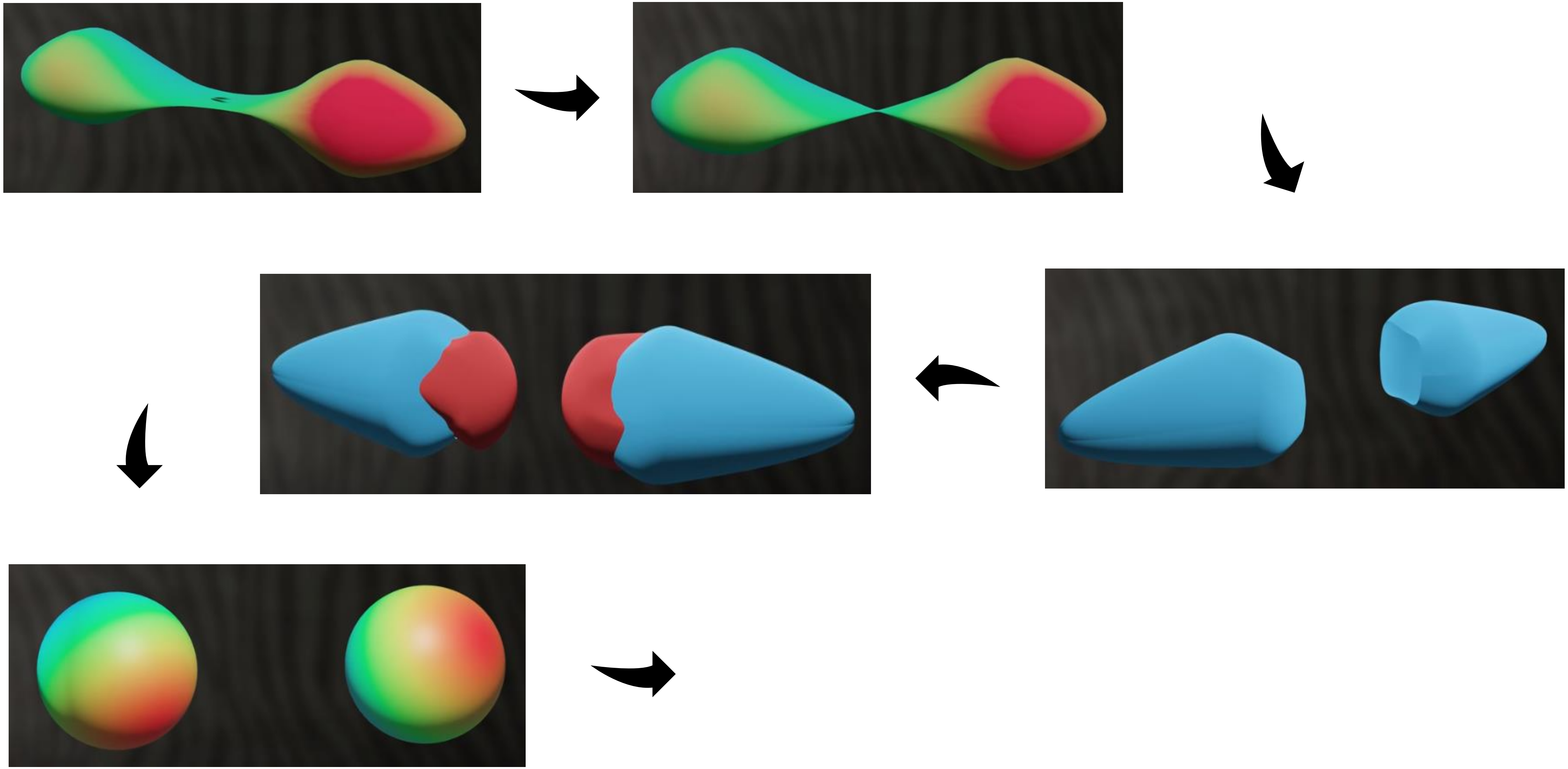
$$g = Metric\ Tensor$$

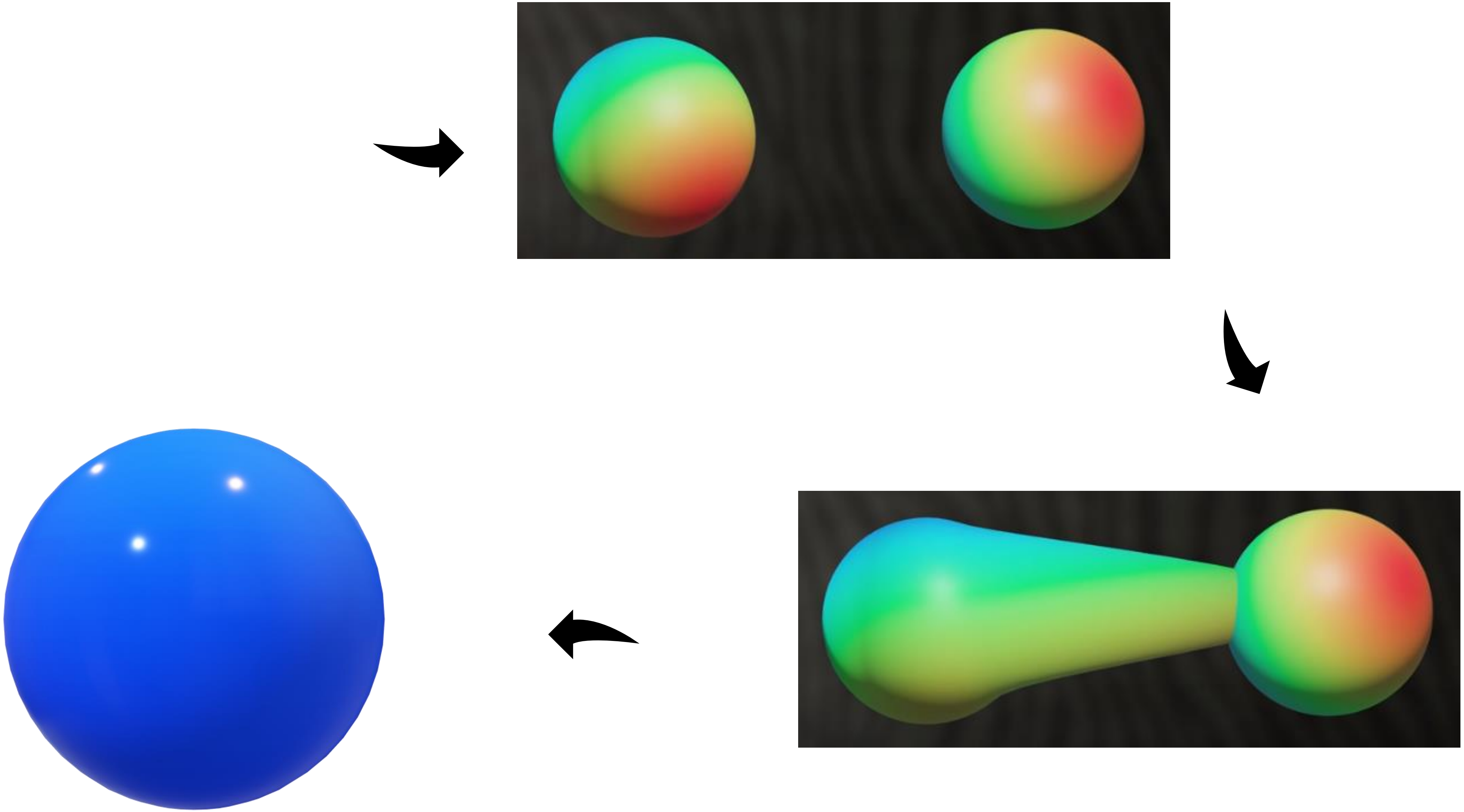
$$R = Ricci\ Curvature$$

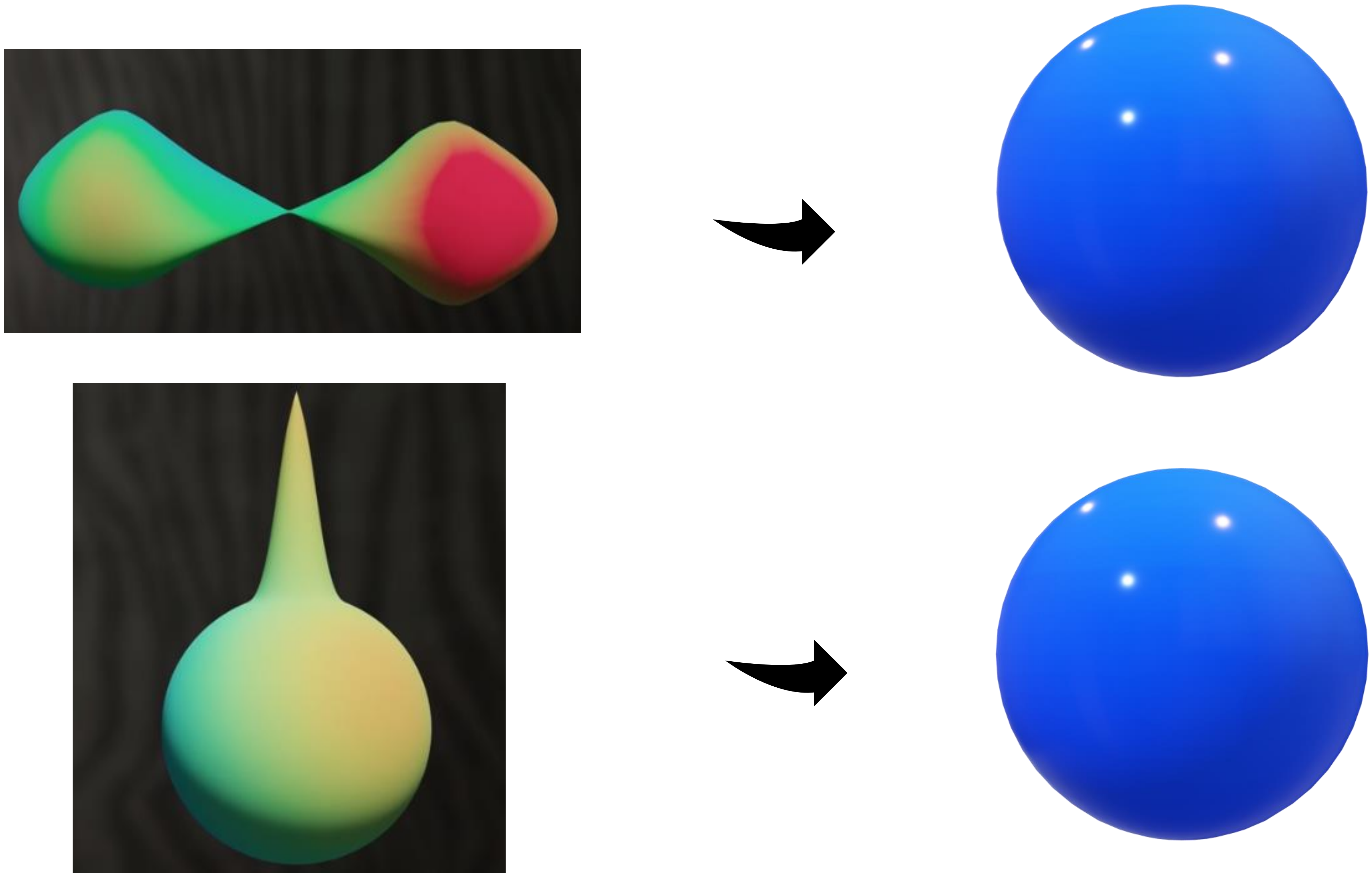


Perelman showed that applying Ricci flow to a sphere reduces it to nothingness. Conversely if Ricci flow is applied to a surface and made it reduced to nothingness, then it must have been a sphere.





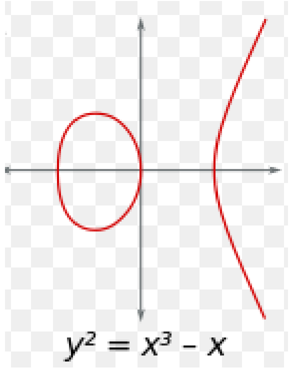


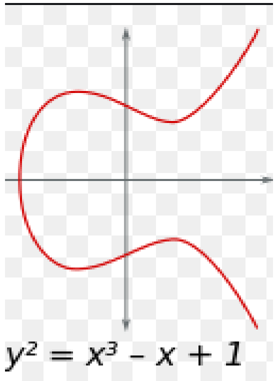


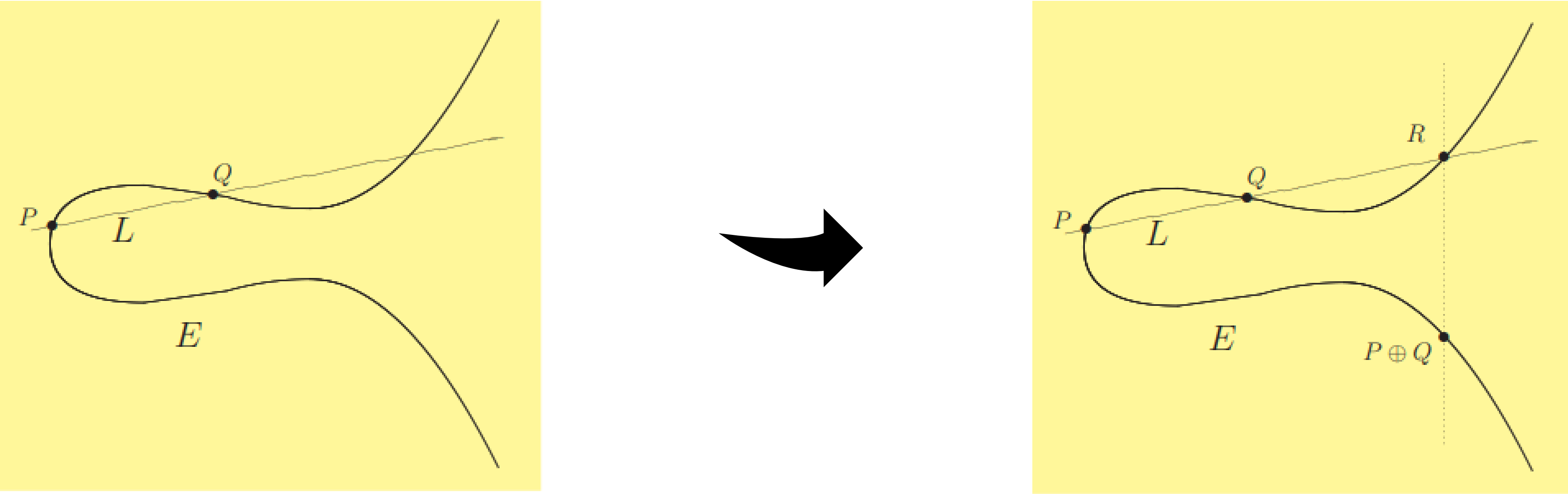
2. The Birch and Swinnerton Dyer Conjecture

An overview to the conjecture.

- The conjecture is named after their pioneers.
- Answer to this conjecture will open doors to ECC (Elliptic Curve Cryptography) and elevate security algorithms to next level.
- The conjecture talks about rational solutions of elliptic curves. Furthermore, it also tells you how many solutions you need in order to generate all the rest of the solutions.
- An equation of an elliptic curve is of the form $y^2 = x^3 + ax + b$ where a and b are constants.
- The conjecture relates Number Theory, Algebraic Geometry, Complex Analysis and Topology.







The rational solutions on an elliptic curve surprisingly forms a finitely generated Abelian group. The group operation is shown in the picture. We can hence express $E(\mathbb{Q})$, the set of rational points on the curve as

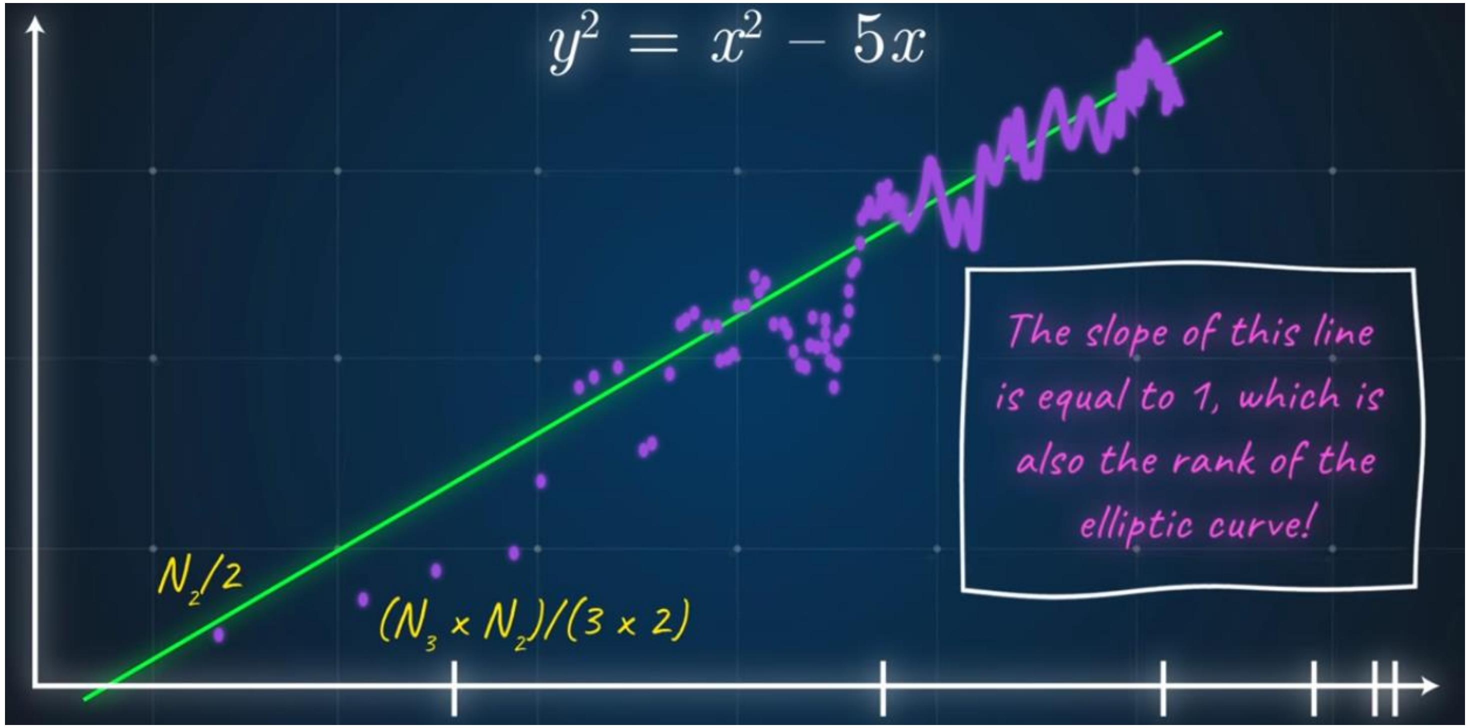
$$E(\mathbb{Q}) = T \oplus \mathbb{Z}^r$$

Where T is the torsion group. A beautiful result by Mazur states that $|T| \leq 16$. Here r is called the rank of the elliptic curve.

Let E be the elliptic curve

$$y^2 = x^2 - 5x$$

We find its solution modulo p. Let N_p denote the number of solution mod p.



p	2	3	5	7	11	13	17	19	23	29
N_p	0	0	4	2	4	5	7	8	10	13

- Birch and Swinnerton observed that the slope of the trend line always equaled the rank of the elliptic curve. From this they conjectured that

$$\prod_{p\leq X}\frac{N_p}{p}\sim c(\log X)^r$$

- This conjecture is also translated in the language of Modular Forms and L-Functions. We denote the Frobenius of an elliptic curve to be

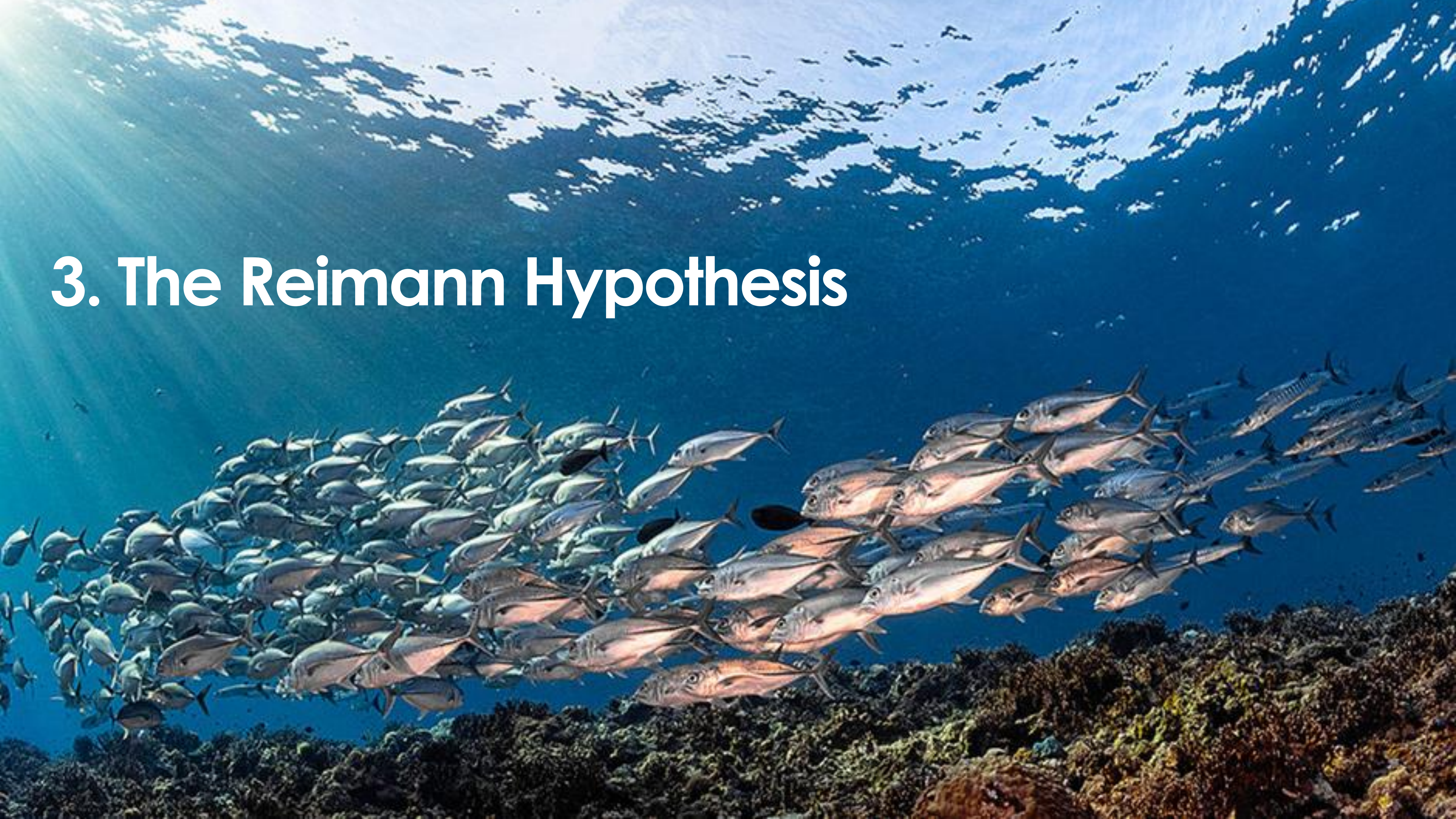
$$a_p = p+1-N_p$$

- Showed that we can correspond an L-Function to any elliptic curve such that

$$L(E,s)=\prod_{prime\ p}\left(1-\frac{a_p}{p^s}+\frac{1}{p^{2s-1}}\right)^{-1} \quad \Longrightarrow \quad L(E,1)=\prod_{prime\ p}\left(1-\frac{a_p}{p}+\frac{1}{p}\right)^{-1}=\prod_{p\leq X}\frac{p}{N_p}$$

Accordingly, the conjecture states $L(E,1)=0 \iff |E(\mathbb{Q})|=\infty$.

3. The Reimann Hypothesis



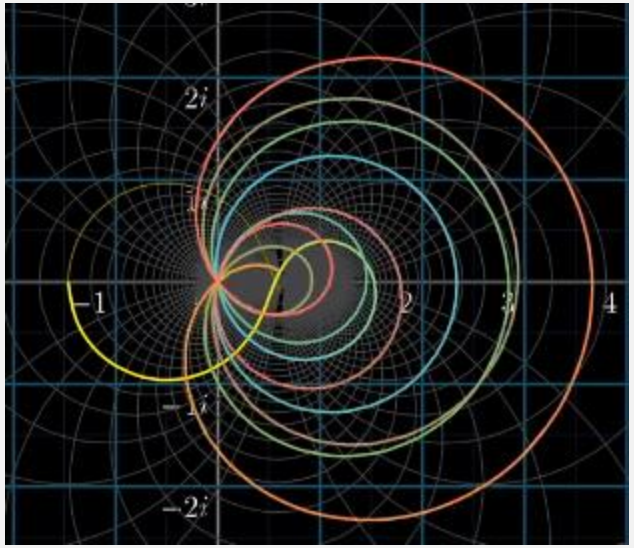
An overview to the conjecture.

- Bernhard Reimann in 1859, stated the conjecture in his paper relating prime counting function. It is one of the most influential paper in analytic number theory.

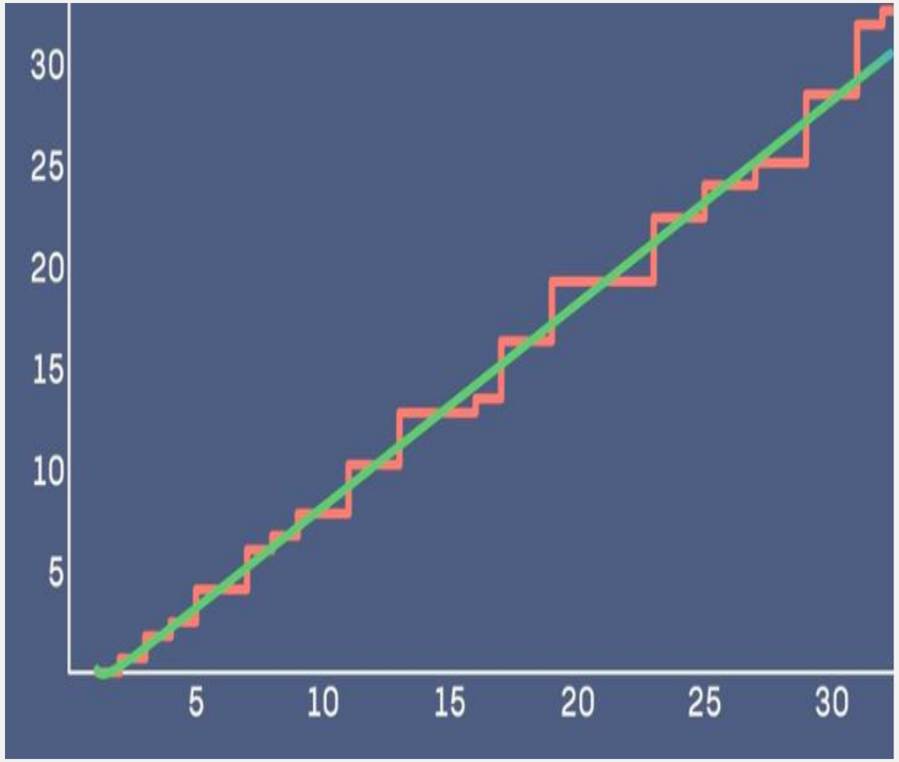
- If the hypothesis is true, it would solve a mystery ,as old as Math itself, that is the mystery of the prime numbers. It will give the distribution of prime numbers.

- The hypothesis uses tools from complex analysis to understand prime numbers. It concerns the Reimann-Zeta function which is

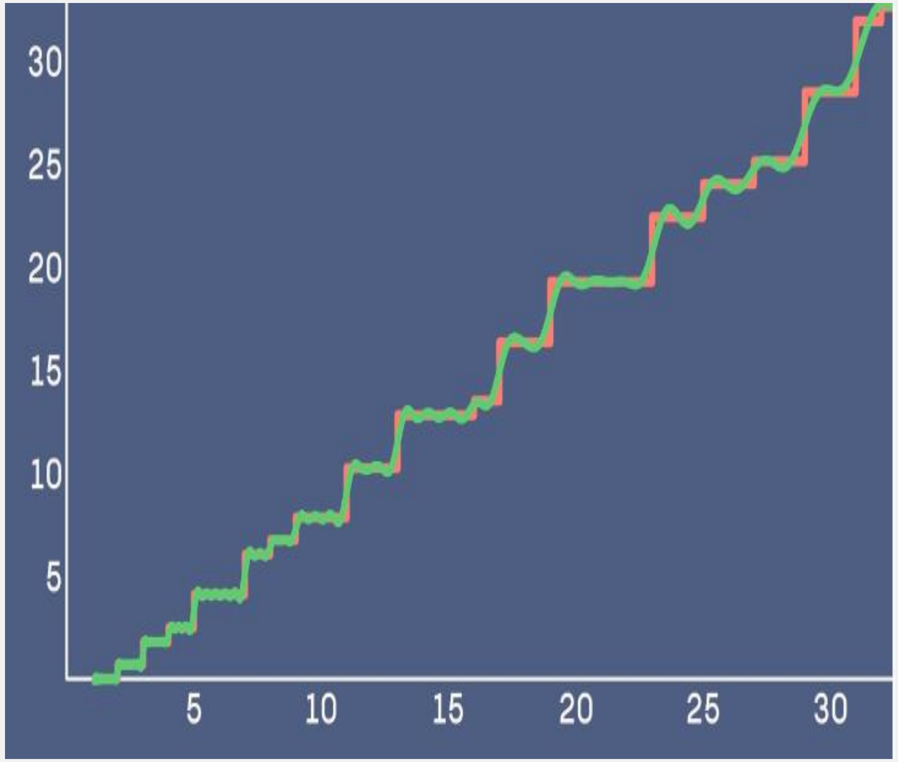
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{prime\ p} (1 - p^{-s})^{-1} \qquad where\ s \in \mathbb{C}$$



- The Hypothesis states that the real part of the non-trivial zeros of the Reimann-zeta function is $\frac{1}{2}$.

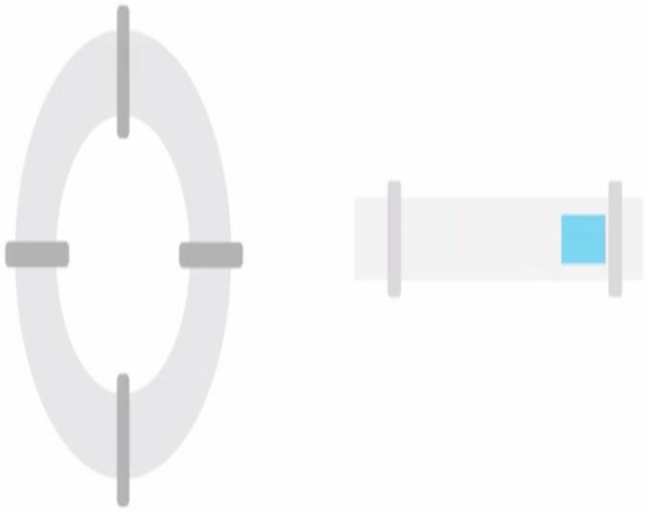


Inserting
10 zeta
zeros



4. The Navier-Stokes Equation

An overview to the problem



- The Navier-Stokes equation models everything that flows in the universe like water, air, honey, gas, smoke etc. Solving it would help us understand any thing that flows.

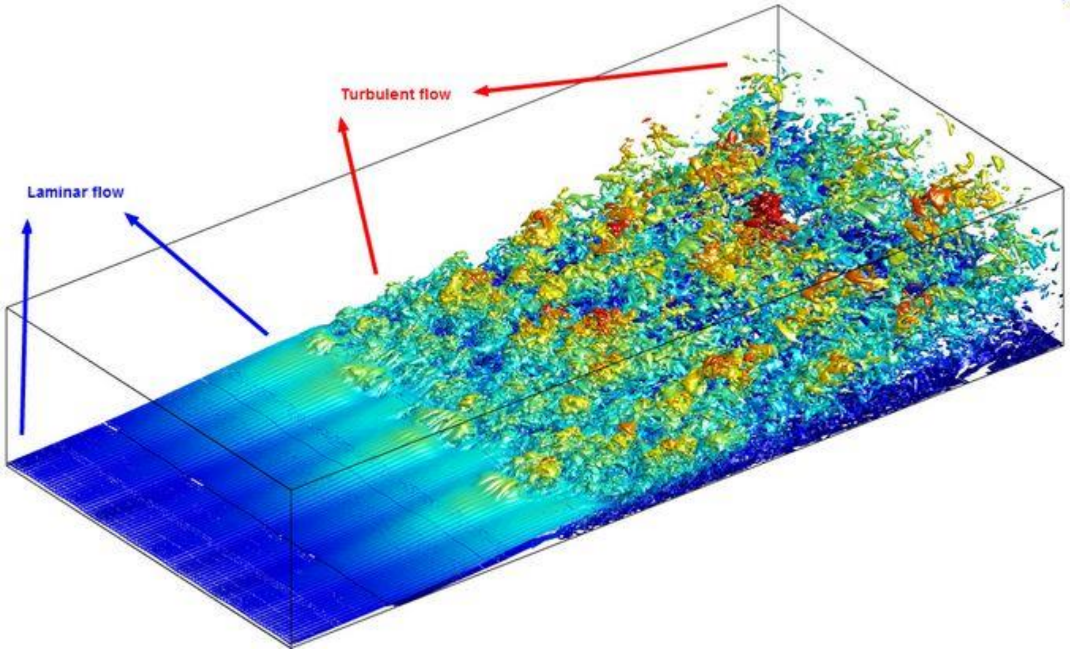
- The Equation is of the following form

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla P + div(T) + f(x,t)$$

Where ρ is the viscosity, $\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right)$ is defined as acceleration as a function of velocity, $-\nabla P$ is the pressure gradient, $div(T)$ is the divergence of stress vector and $f(x,t)$ are the other forces.

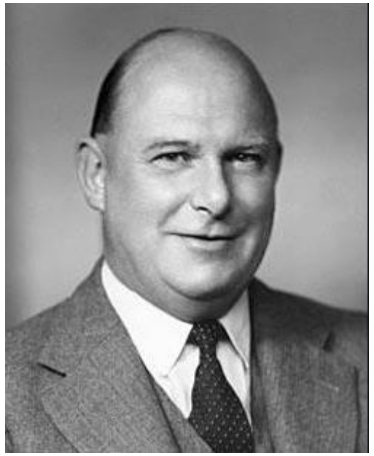
- Under special situation, it is easy to solve the equation, but we aren't able yet to solve it for any general case.

- The million dollars question is that for fluids in 3-dimensions, can we determine if the solution to this equation exists? And if they exist, are they smooth or differentiable everywhere?

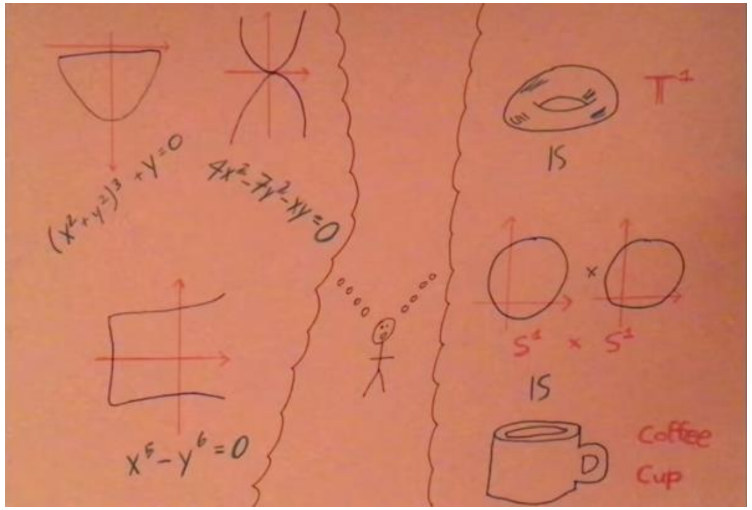

Turbulent flow
Laminar flow

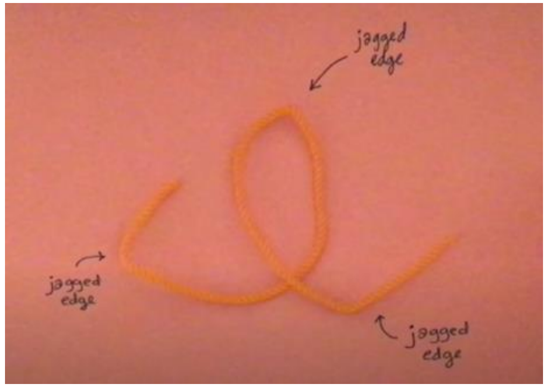
5. The Hodge Conjecture

An overview to the conjecture.



- The conjecture was given by W.V.D Hodge in the 20th century.
- Solving this conjecture will help us understand the true abstract nature of Mathematics. More specifically, it will act as a bridge of exchanging information between Algebraic Geometry and Topology.
- It is also widely said that Mathematics is still not ready to attack this problem (just like the Hailstone sequence).
- Roughly Roughly saying, the conjecture says given a random shape, when is it homeomorphic to a shape described by polynomials? Suppose you have a string with jagged edges, the conjecture says you can squish it into something with smooth edges.
- Micheal Freedman gave a counterexample by constructing a shape that cannot be expressed by polynomials. Hence, we then have the revised Hodge conjecture stating that **find one condition that ensure a shape can be molded into an algebraic set.**

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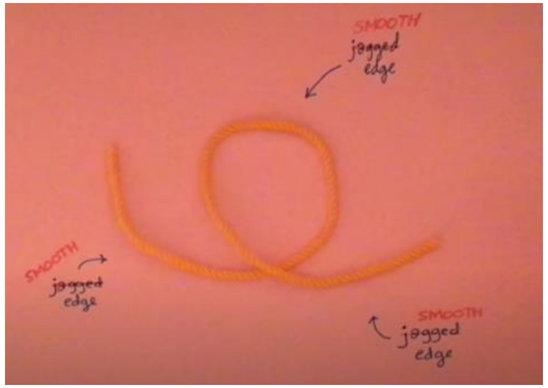
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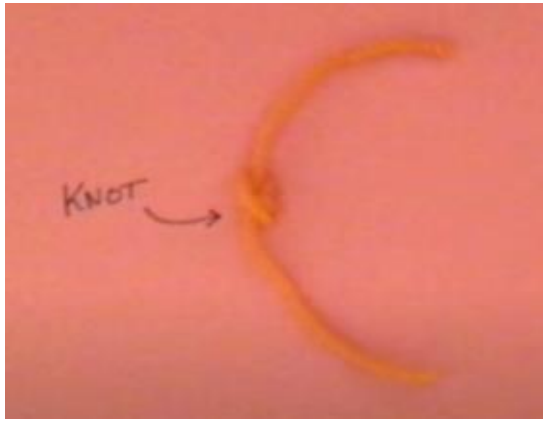
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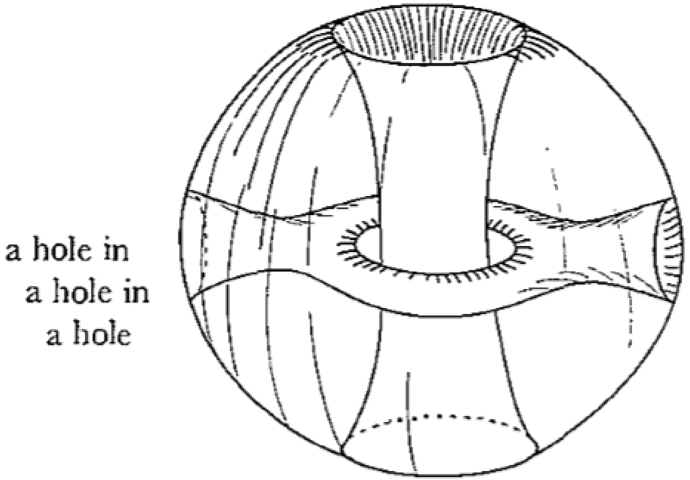

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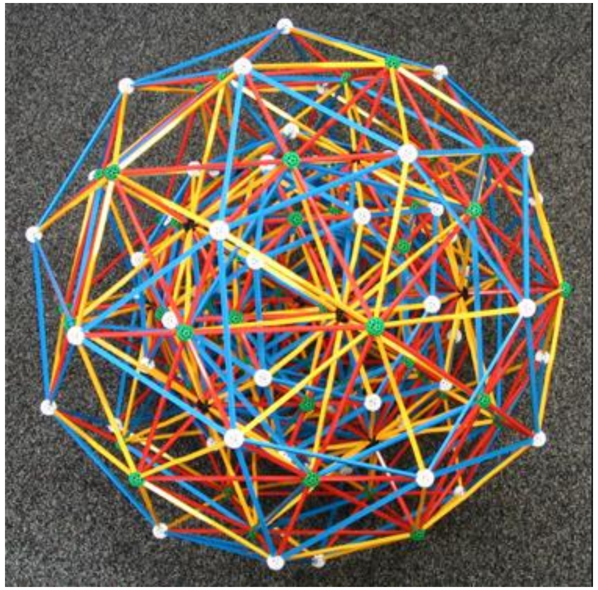
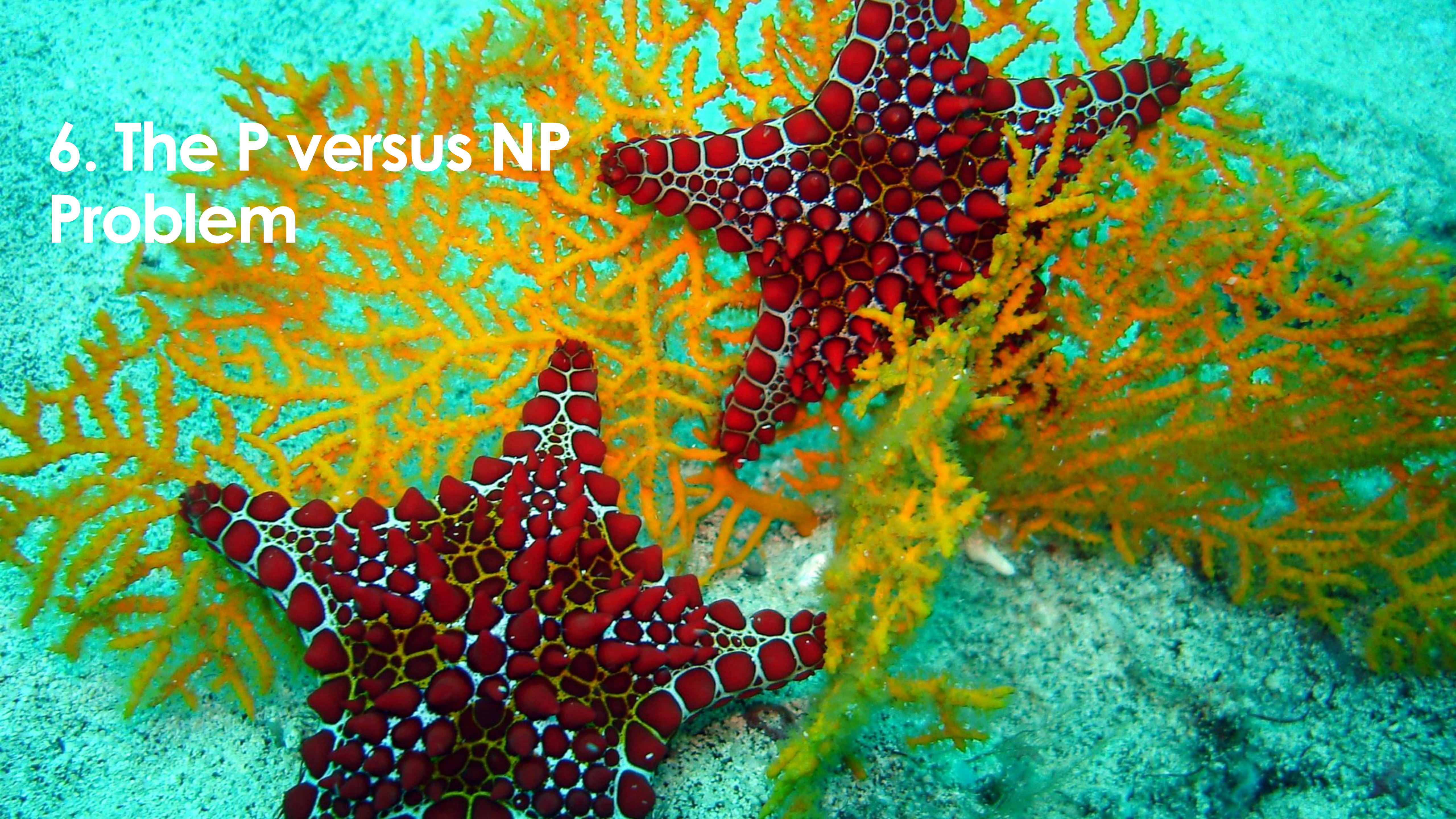


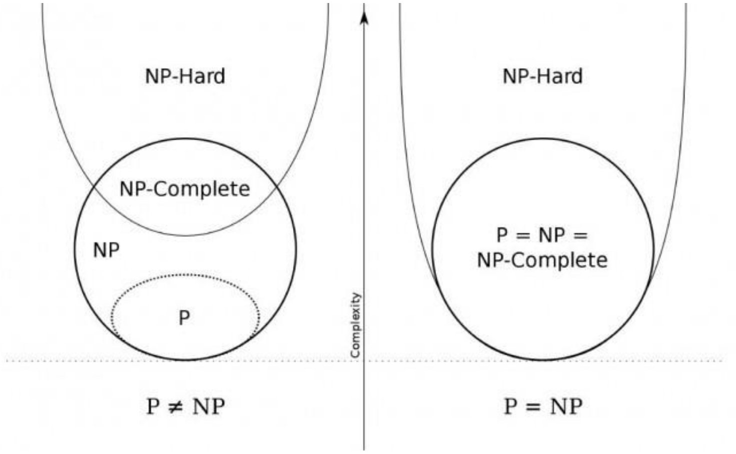
There is no polynomial equation that can ever describe it. It is called the Freedman E8 manifold living in 4 dimensions.

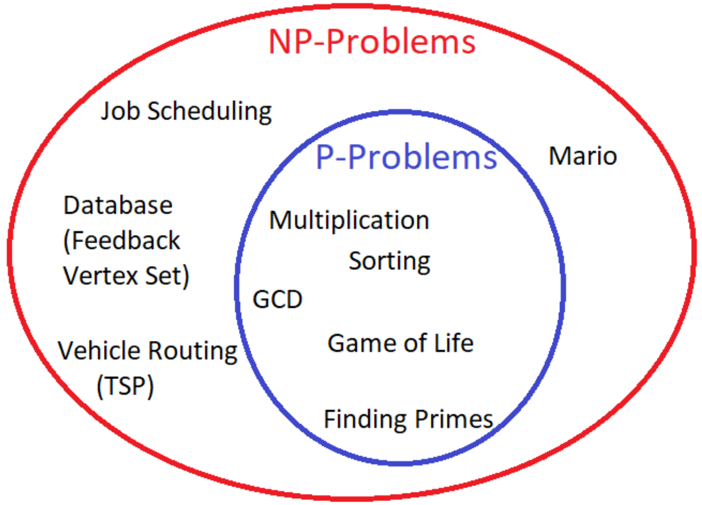
6. The P versus NP Problem



An overview to the problem.

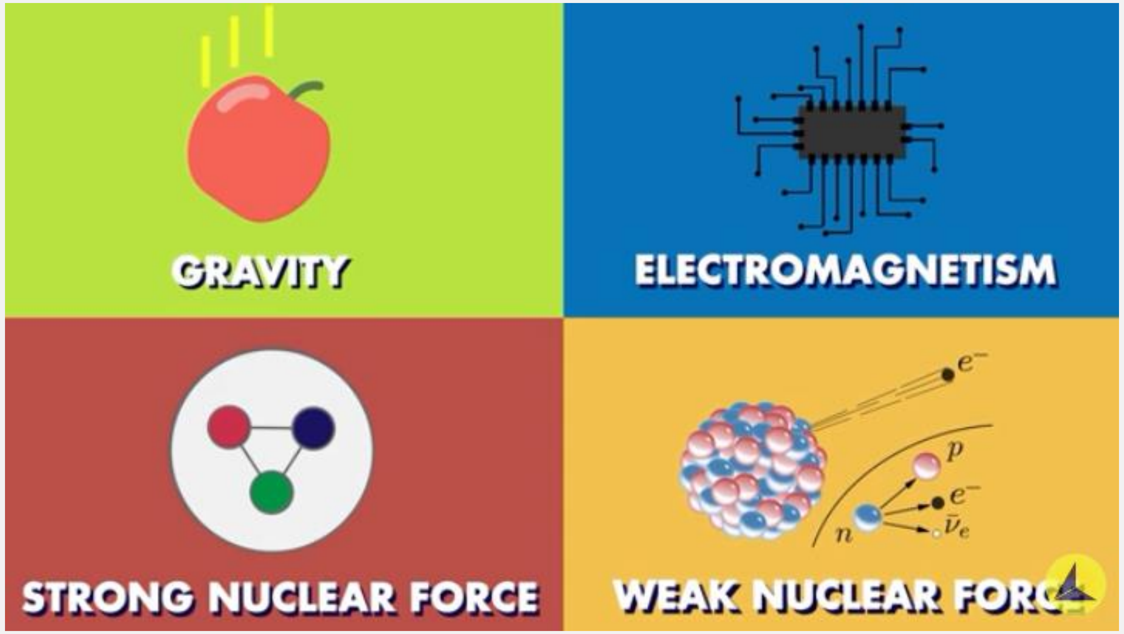
- This problem is of fundamental importance for computer scientist.
- P stands for the class of problems that can be solved in polynomial time or less.
- NP stands for class of problems that can be solved in exponential time or less. This class contains the world's most complex problems.
- The question is that Is **P=NP** ?
- If **P=NP** then, it will imply that no matter how tough the problem seems, there will always exists a simple algorithm, which could solve it in polynomial time.
- If **P≠NP** then things would be very problematic as computation of some question would be very time expensive and we might become helpless to solve some complex algorithms.





7. Quantum Yang-Mills Theory

An overview to the problem.



- This problem is relating Quantum Field Theory.
- Solving this problem can lay out Mathematical grounds for the Grand Unified Theory.
- To make connections between these 4 forces, one must observe is on the quantum level.
- To talk about Quantum field theory, one must rely upon Gauge theory which takes in account for the symmetry present in nature.
- It was later on observed that a successful Gauge Theory, is the theory which also satisfies the Yang-Mills theory which introduce non-abelian Lie Groups into the picture.
- The problem is that prove that for every Gauge group, there exists a Yang-Mills theory in a 4D Hilbert space with a finite mass gap.

> We know very little, and yet it is astonishing that we know so much, and still more astonishing that with so little knowledge can give us so much power.

> Mathematics has an evolving infinite adaptable boundary whose center is everywhere. There is no beginning, there is no end. But we are soul. We have eternity to in-gauge. The gradient of potential will is desire.

References

- The Millennium Prize Problems by J. Clarson, A. Jaffe and A. Wiles
- Channels on youtube: Kinertia, Aleph 0, 3Blue1Brown, MIT, Clay Mathematical Institute, Numberphile.


Thank You

23 Feb, 2021