Unipotent Numerical Semigroups

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Abstract

The poster intends to introduce the generalization of the theory of numerical semigroups to the theory of unipotent numerical semigroups. We were able to naturally extend the associated invariants to the unipotent setting which enabled us to generalize the Wilf conjecture and classify the semigroups accordingly. Furthermore, we define ideals, blowups and unipotent Arf semigroups and were able to prove that the blowup and unipotent Arf closure commutes.

Numerical Semigroup

Let $\mathcal{S}$ be a complement finite submonoid of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (a.k.a numerical semigroup). The associated invariants are:

- Conductor of $\mathcal{S}$: $\mathtt{c}(\mathcal{S}) := \max\{\mathbb{N} \backslash \mathcal{S}\} = \min_{c \in \mathbb{N}}\{c + \mathbb{N} \subseteq \mathcal{S}\}$.
- Generating number of $\mathcal{S}$: $\mathtt{r}(\mathcal{S}) = \mathtt{c}(\mathcal{S}) + 1$.
- Cardinality of sporadic elements of $\mathcal{S}$: $\mathtt{n}(\mathcal{S}) := |\{s \in \mathcal{S} : s < \mathtt{c}(\mathcal{S})\}|$.
- Cardinality of minimum generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$ (excluding 0).
- Genus of $\mathcal{S}$: $\mathtt{g}(\mathcal{S}) = |\mathbb{N} \backslash \mathcal{S}|$.

Example

Let $\mathcal{S} = \{0, 4, 7, 8, 9, 10, ...\} = \langle 4, 7, 9, 10\rangle$. Here $\mathtt{c}(\mathcal{S}) = 6$, $\mathtt{n}(\mathcal{S}) = 2$ and $\mathtt{e}(\mathcal{S}) = 4$.

Unipotent Numerical Semigroups

Let $\mathbf{U} = \mathbf{U}(n, \mathbb{N}_0)$ denote $n \times n$ unipotent upper triangular matrices with entries in $\mathbb{N}_0$. Also, let $\mathbf{P} = \mathbf{P}(n, \mathbb{N}_0)$ denote the maximal commutative submonoid in $\mathbf{U}$, which happens to be isomorphic to $\mathbb{N}_0^{n-1}$.

$$\mathbf{U} = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \cong \mathbb{N}_0^{n-1}$$

where all entries are from $\mathbb{N}_0$. Subsequently, we define $\mathbf{U}_k$ (similarly $\mathbf{P}_k$, $\mathcal{M}_k$) as

$$\mathbf{U}_k := \{(x_{ij})_{1 \le i,j \le n} \in \mathbf{U}(n, \mathbb{N}_0) : \max_{1 \le i,j \le n} x_{ij} \ge k\}$$

Suppose $\mathcal{M} \subseteq \mathbf{U}$ be a finitely generated submonoid. The submonoid $\mathcal{S} \subseteq \mathcal{M}$ is said to be a **unipotent numerical semigroup** if $|\mathcal{M} \backslash \mathcal{S}|$ is finite. We now define invariants corresponding to $\mathcal{S}$.

- Generating number of $\mathcal{S}$: $\mathtt{r}_{\mathcal{M}}(\mathcal{S}) := \min\{k \in \mathbb{N} : \mathcal{M}_k \subseteq \mathcal{S}\}$.
- Conductor of $\mathcal{S}$: $\mathtt{c}_{\mathcal{M}}(\mathcal{S}) = \mathtt{r}_{\mathcal{M}}(\mathcal{S})^n$.
- Embedding dimension of $\mathcal{S}$: $\mathtt{e}(\mathcal{S}) := \min\{|\mathcal{G}| : \langle \mathcal{G} \rangle \text{ generates } \mathcal{S}\}$.
- Dimension of $\mathcal{M}$: $\mathtt{d}_{\mathcal{M}} := \mathtt{e}(\mathcal{M})$
- Genus of $\mathcal{S}$: $\mathtt{g}_{\mathcal{M}}(\mathcal{S}) = |\mathcal{M} \backslash \mathcal{S}|$.
- Cardinality of sporadic elements of $\mathcal{S}$: $\mathtt{n}_{\mathcal{M}}(\mathcal{S}) = |\mathcal{S} \backslash \mathcal{M}_{\mathtt{r}_{\mathcal{M}}(\mathcal{S})}| + 1$.

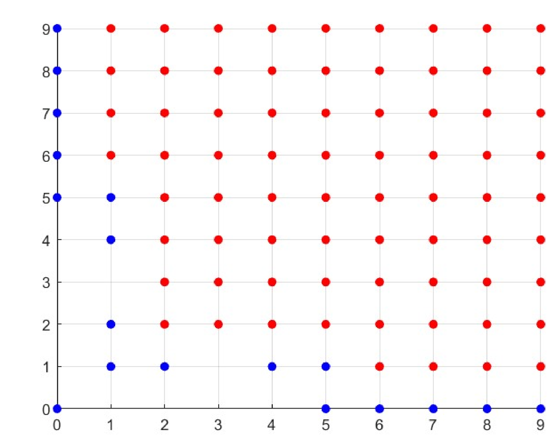
The Unipotent Wilf Conjecture

Conjecture: Let $\mathcal{M}$ be a finitely generated submonoid of $\mathbf{U}(n, \mathbb{N}_0)$ and $\mathcal{S}$ be a unipotent numerical semigroup of $\mathcal{M}$, then we have

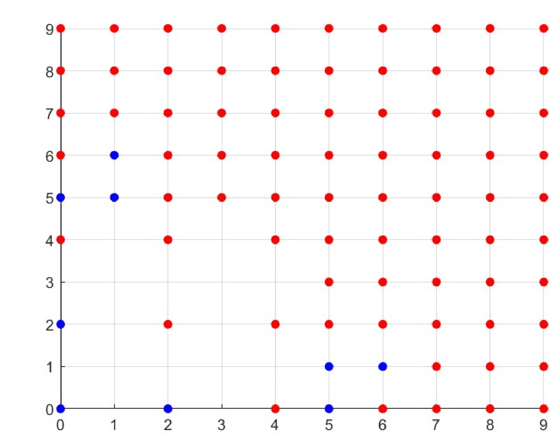
$$\mathtt{d}_{\mathcal{M}} \mathtt{c}_{\mathcal{M}}(\mathcal{S}) \le \mathtt{e}(\mathcal{S}) \mathtt{n}(\mathcal{S}).$$

In [2], we have shown that this is a reasonable inequality to consider. It does attain equality in some cases. We have introduced two sub-families of unipotent numerical semigroup in the paper, thick and thin, for which the conjecture holds.

Examples of the conjecture



(a) $\mathtt{c}(\mathcal{S}) = 25,\ \mathtt{e}(\mathcal{S}) = 17,\ \mathtt{n}(\mathcal{S}) = 15$



(b) $\mathtt{c}(\mathcal{S}) = 25,\ \mathtt{e}(\mathcal{S}) = 8,\ \mathtt{n}(\mathcal{S}) = 9$

Ideals: Blowup and Multiplicity Sequence

Let $\mathcal{M} = \mathcal{M}_{\mathbb{N}_0} = \mathbf{P}(n, \mathbb{N}_0)$ and $\mathcal{S}$ be a unipotent numerical semigroup of $\mathcal{M}$. We define $\mathcal{M}_{\mathbb{Z}} = \mathbf{P}(n, \mathbb{Z})$.

Ideals:

The subset $\mathcal{I} \subseteq \mathcal{M}_{\mathbb{Z}}$ is said to be a **relative ideal** to $\mathcal{S}$ if $\mathcal{I}$ satisfies the following:
1. $\mathcal{I}\mathcal{S} \subseteq \mathcal{I}$.
2. there exists $s \in \mathcal{S}$ such that $s\mathcal{I} \subseteq \mathcal{S}$.

In addition, if $\mathcal{I} \subseteq \mathcal{M}$, then $\mathcal{I}$ is called an **ideal** of $\mathcal{S}$. We will assume that $\mathcal{I} \neq \mathcal{S}$

Now if $\mathcal{I}$ and $\mathcal{J}$ are relative ideals of $\mathcal{S}$ then

$$\mathcal{I} + \mathcal{J} = \mathcal{I}\mathcal{J} := \{IJ : I \in \mathcal{I},\ J \in \mathcal{J}\}, \quad \text{and} \quad \mathcal{I} - \mathcal{J} := \{A \in \mathcal{M}_{\mathbb{Z}} : A\mathcal{J} \subseteq \mathcal{I}\}$$

are also relative ideals of $\mathcal{S}$.

Blowup:

Now for any such set, one can define the "minimal" element of $\mathcal{I}$ as

$$\mathtt{m}(\mathcal{I}) := \{A \in \mathcal{I}^* : ||A|| \text{ is minimum }\}.$$

If $\mathcal{I}$ is an ideal of $\mathcal{S}$ such that $|\mathtt{m}(\mathcal{I})| = 1$, then there exists $h \in \mathbb{N}$ for which we have

$$\mathcal{I}^h - \mathtt{m}(\mathcal{I})^h = \mathtt{m}(\mathcal{I})^{h+1} - \mathcal{I}^{h+1}$$

We denote the smallest such h by $\mathtt{r}(\mathcal{I})$ and call it the **reduction number** of $\mathcal{I}$. With all this, we define the **blowup** of $\mathcal{I}$ to be

$$\mathtt{B}(\mathcal{I}) := \mathcal{I}^{\mathtt{r}(\mathcal{I})} - \mathtt{m}(\mathcal{I})^{\mathtt{r}(\mathcal{I})}$$

Blowup of $\mathcal{S}$ is defined as $\mathtt{B}(\mathcal{S}) := \mathtt{B}(\mathcal{S}^*)$ where $\mathcal{S}^* := \mathcal{S} \backslash \mathbf{0}$ is the maximal ideal contained in $\mathcal{S}$. For any ideal $\mathcal{I}$ of $\mathcal{S}$, we always have $\mathcal{S} \subseteq \mathtt{B}(\mathcal{I}) \subseteq \mathcal{M}$.

Multiplicity Sequence:

Let $\mathcal{S}^{(0)} = \mathcal{S}$ and $\mathcal{S}^{(i)} = \mathtt{B}(\mathcal{S}^{(i-1)})$ for $i \in \mathbb{N}$. We then have a chain

$$\mathcal{S} = \mathcal{S}^{(0)} \subseteq \mathcal{S}^{(1)} \subseteq \cdots \subseteq \mathcal{S}^{(q)} = \mathcal{M}$$

As $|\mathcal{M} \backslash \mathcal{S}|$ is finite, the chain eventually stabilizes, and we can obtain a sequence $(\mathtt{d}_0, \mathtt{d}_1, \ldots, \mathtt{d}_q)$, where $\mathtt{d}_i = \text{floor}(\min_{A \in \mathtt{m}(\mathcal{S}^i)} ||A||)$ and $||A||$ is the Euclidean metric on vector in $\mathbb{N}^{n-1}$. We call this the **multiplicity sequence** of $\mathcal{S}$. This is a non-increasing sequence that ends at 1.

Example

Let $\mathcal{S} = \langle \mathbf{0}, (1,2), (2,1), \mathbf{P}_3 \rangle \subseteq \mathcal{M} = \mathbf{P}(3, \mathbb{N}_0)$. Then as $\mathtt{r}(\mathcal{S}^*) = 1$ we have the chain

$$\mathcal{S} \subseteq \langle \mathcal{S}, (1,1), (0,2), (2,0) \rangle \subseteq \langle \mathcal{S}^{(1)}, (1,0), (0,1) \rangle = \mathcal{M}$$

giving the multiplicity sequence $(2, 1, 1)$. Note that $|\mathtt{m}(\mathcal{S}^*)| = 2$

Unipotent Arf semigroups

We start by giving the following definition. We assume $\mathcal{M} = \mathbf{P}$.

Definition: Let $\mathcal{S}$ be a unipotent numerical semigroup and $\mathcal{I}$ be an ideal of $\mathcal{S}$. Then $\mathcal{I}$ is said to be **integrally closed** if

$$\mathcal{I} = \mathcal{S}(A) := (A\mathcal{M}) \cap \mathcal{S}$$

for some $A \in \mathcal{S}$. □

We now define **Unipotent Arf semigroups**.

Definition: A unipotent numerical semigroup $\mathcal{S}$ is said to be ***Arf*** if every integrally closed ideal $\mathcal{I}$ of $\mathcal{S}$ is stable i.e. $\mathtt{r}(\mathcal{I}) = 1$. The ***Arf* closure** of $\mathcal{S}$ is the smallest unipotent *Arf* semigroup containing $\mathcal{S}$. □

Blowup and Arf closure commutes:

If $\mathcal{S}$ is a unipotent numerical semigroup such that $|\mathtt{m}(\mathcal{S}^*)| = 1$, then

$$\mathtt{Arf}(\mathtt{B}(\mathcal{S}^*)) = \mathtt{B}(\mathtt{Arf}(\mathcal{S})).$$

■

Example

Let $\mathcal{S} = \langle (1,2), \mathbf{P}_3 \rangle \subseteq \mathcal{M} = \mathbf{P}(3, \mathbb{N}_0)$. We have $\mathtt{m}(\mathcal{S}^*) = (1,2)$, $\mathtt{r}(\mathcal{S}^*) = 2$, $\mathtt{B}(\mathcal{S}^*) = \mathcal{M}$ which is Arf. On the other hand, $\mathtt{Arf}(\mathcal{S}) \neq \mathcal{S}$ as $\mathcal{I} = \mathcal{S}((2,0))$ is not stable i.e. $\mathtt{r}(\mathcal{I}) = 2$. By calculation, we have $\mathtt{Arf}(\mathcal{S}) = \langle (1,1), \mathbf{P}_2 \rangle$ and $\mathtt{B}(\mathtt{Arf}(\mathcal{S})) = \mathcal{M}$.

Ultimate goal

Our ultimate goal is to connect our idea with algebraic coding theory and generate different families of codes or determine sharper bounds on the minimum Hamming distance of the codes. This is motivated by the following idea:

Let X be a surface of genus $g \ge 1$. One can extract a numerical semigroup $\mathcal{S}$, from the set of order of poles at a point $P \in X$ of regular functions on X. Such numerical semigroup is also known as the Weierstrass semigroup. It is well known that we have

$$\mathtt{g}(\mathcal{S}) = g$$

There have been generalizations of Weierstrass semigroup to multiple points on the curve in [1] and [3]. In [4], the author was able to relate the genus of the Weierstrass semigroup of pair of points on the curve $\mathtt{g}(\mathcal{S})$ and the genus of the curve g by the following inequality

$$\binom{g+2}{2} - 1 \le \mathtt{g}(\mathcal{S}) \le \binom{g+2}{2} - 1 - g + g^2$$

We aim to offer the same analysis tool on curves using the idea of unipotent numerical monoids.

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