Blowup and Arf Unipotent Monoids

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Numerical Semigroups

We assume $\mathbb{N} = \{0, 1, 2, 3, \rightarrow\}$ throughout the talk.

Definition

A subset $\mathcal{S} \subseteq \mathbb{N}$ is a numerical semigroup if

- $0 \in \mathcal{S}$.
- If $a, b \in \mathcal{S}$ then $a + b \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbb{N}$ is finite.

Invariants

- The *generating number* of $\mathcal{S}$ is the smallest integer $\mathtt{k}$ such that $\{\mathtt{k}+1, \mathtt{k}+2, \cdots\} \subseteq \mathcal{S}$.
- The *multiplicity* of $\mathcal{S}$ is the smallest nonzero element in $\mathcal{S}$.

Example

Let $\mathcal{S} = \{0, 6, 9, 12, 13, 14, \rightarrow\} = \langle 6, 9, 13, 14, 16, 17\rangle$.

Unipotent Numerical Monoids

Definition (Can, S. 23)

Let

$$\mathbf{U}(n,\mathbb{N}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{ij}\}_{1<i<j<n} \in \mathbb{N} \right\}$$

We fix a finitely generated monoid $\mathbf{G} \subseteq \mathbf{U}(n,\mathbb{N})$ (takes place as $\mathbb{N}$).
A subset $\mathcal{S} \subseteq \mathbf{G}$ is a *unipotent numerical semigroup* if

- $\mathbf{1}_n \in \mathcal{S}$.
- If $A, B \in \mathcal{S}$ then $AB \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbf{G}$ is finite.

If $\mathbf{G} = \mathbf{P}(n, \mathbb{N})$ where

$$\mathbf{P}(n,\mathbb{N}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n+1)\times(n+1)} : \{x_{1j}\} \in \mathbb{N} \right\} \quad (\cong \mathbb{N}^n)$$

Notation

For simplicity, if $\mathbf{G} = \mathbf{P}(n, \mathbb{N}$, we denote its element in n-tuple format

$$(x_1, \ldots, x_n)$$

Example (k-th Fundamental monoid)

Let

$$\mathbf{P}_k(n) := \{(x_{1j})_{1<j\leq n} \in \mathbf{P}(n) : x_{1j} \geq k \text{ for some } 1 > j \leq n\} \subseteq \mathbf{P}(n).$$

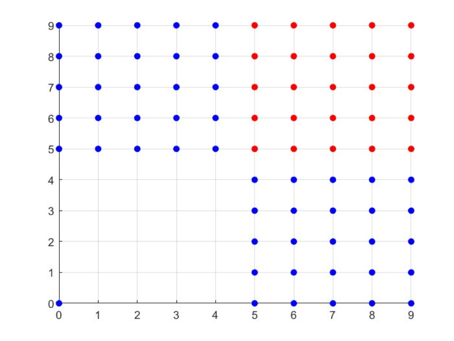


Figure: This is $\mathbf{P}_5(2)$

Example

Let $\mathcal{S} \subseteq \mathbf{P}(2)$ and consider $\mathcal{S}$ plotted as

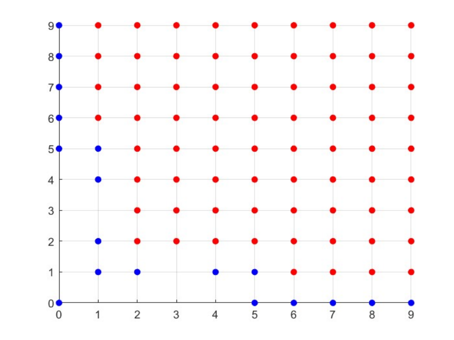


Figure: $\mathcal{S} = \langle (1,1), (2,1), (1,2), (4,1), (1,4), \mathbf{P}_5 \rangle$

Notation

1. Let $\mathbf{G_U}$ denote $\mathbf{U}(n, \mathbb{N})$ and $\mathbf{G_P}$ denote $\mathbf{P}(n, \mathbb{N})$. Collectively we denote them by $\mathbf{G}$.
2. We let $\mathbf{G}_k$ denote the corresponding k-th Fundamental monoid in $\mathbf{G_U}$ or $\mathbf{G_P}$.
3. An asterisk on a set denote the set minus the identity element e.g. $\mathbf{G}^* = \mathbf{G} \backslash \mathbf{1}_n$ where $\mathbf{1}_n$ denote the $n \times n$ identity matrix.
4. $\mathcal{S}$ will denote an arbitrary Unipotent Numerical Monoid in $\mathbf{G}$.

Partial Orders

Partial orders

Let A and B be arbitrary elements in $\mathbf{G}$, then

- $A \leq_{\mathbf{G},r} B \iff BA^{-1} \in \mathbf{G}$.
- $A \leq_{\mathbf{G},l} B \iff A^{-1}B \in \mathbf{G}$.
- $A \leq_{\mathbf{G},t} B \iff \{A^{-1}B, BA^{-1}\} \subseteq \mathbf{G}$.

Invariants

Generating number and Multiplicities

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$.

- Generating number $\mathtt{r}(\mathcal{S}) = \min\{k \in \mathbb{N} : \mathbf{G}_k \subseteq \mathcal{S}\}$.
- Two-sided Multiplicity of $\mathcal{S}$ is

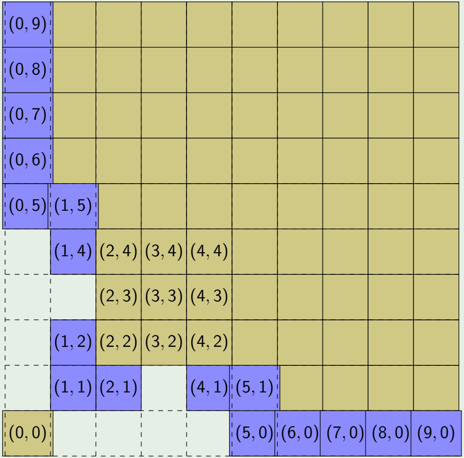
$$\mathtt{m}_t(\mathcal{S}) := \{A \in \mathcal{S}^* \mid \ \forall\ B \in \mathcal{S}^*, \text{ either } A \leq_{\mathbf{G},t} B \text{ or } A \nleq_{\mathbf{G},t} B\}$$

- (Left,Right) Multiplicity of $\mathcal{S}$ is

$$\mathtt{m}_{l,r}(\mathcal{S}) := \{A \in \mathcal{S}^* \mid \ \forall\ B \in \mathcal{S}^*, \text{ either } A \leq_{\mathbf{G},(l,r)} B \text{ or } A \nleq_{\mathbf{G},(l,r)} B\}$$

Example

Let $\mathcal{S} = \langle (1,1),(1,2),(1,4),(2,1),(4,1)\rangle \sqcup \mathbf{G}_5$ in $\mathbf{G} = \mathbf{G_P} = \mathbf{P}(3,\mathbb{N})$.



Generating number $\mathtt{k} = 5$ and $\mathtt{m}_{(r,l,t)}(\mathcal{S}) = \{(1,1),(5,0),(0,5)\}$

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Relative Ideals

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid in $\mathbf{G}$. We define three types of relative ideals of $\mathcal{S}$ in $\mathbf{G}$:

1. The subset $\mathcal{I} \subseteq \mathbf{G}$ is said to be a relative two-sided ideal of $\mathcal{S}$ if $\{\mathcal{I}\mathcal{S}, \mathcal{S}\mathcal{I}\} \subseteq \mathcal{I}$.
2. The subset $\mathcal{I} \subseteq \mathbf{G}$ is said to be a relative right ideal of $\mathcal{S}$ if $\mathcal{I}\mathcal{S} \subseteq \mathcal{I}$.
3. The subset $\mathcal{I} \subseteq \mathbf{G}$ is said to be a relative left ideal of $\mathcal{S}$ if $\mathcal{S}\mathcal{I} \subseteq \mathcal{I}$.

Collectively, we call them relative ideal of $\mathcal{S}$.

Set of generators

Definition

Let $\mathcal{I}$ be a relative ideal of $\mathcal{S}$ in $\mathbf{G}$.

1. If $\mathcal{I}$ is a relative right ideal, then the set $\mathcal{E} \subseteq \mathcal{I}$ generates $\mathcal{I}$ if
$$\mathcal{I} = \mathcal{E}\mathcal{S}.$$
2. If $\mathcal{I}$ is a relative left ideal, then the set $\mathcal{E} \subseteq \mathcal{I}$ generates $\mathcal{I}$ if
$$\mathcal{I} = \mathcal{S}\mathcal{E}.$$
3. If $\mathcal{I}$ is a relative right ideal, then the set $\mathcal{E} \subseteq \mathcal{I}$ generates $\mathcal{I}$ if
$$\mathcal{I} = \{\mathcal{E}\mathcal{S} \cup \mathcal{S}\mathcal{E}\}.$$

Main Result

Theorem (Can, S.)

Let $\mathcal{I}$ be a relative ideal of unipotent numerical monoid $\mathcal{S}$ in $\mathbf{G}$, then $\mathcal{I}$ is finitely generated.

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Operations

Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be relative ideals of $\mathcal{S}$ in $\mathbf{G}$. Then

$$\mathcal{I} + \mathcal{J} := \{AB \mid A, B \in \mathcal{I} \cup \mathcal{J}\}.$$

We define the operation $\mathcal{I} - \mathcal{J}$ as follows:

1. If $\mathcal{I}$ and $\mathcal{J}$ are relative two-sided ideals then

$$\mathcal{I} -_t \mathcal{J} := \{A \in \mathbf{G} \mid \{\mathcal{J}A, A\mathcal{J}\} \subseteq \mathcal{I}\}.$$

2. If $\mathcal{I}$ and $\mathcal{J}$ are relative right ideals then

$$\mathcal{I} -_r \mathcal{J} := \{A \in \mathbf{G} \mid \mathcal{J}A \subseteq \mathcal{I}\}.$$

3. If $\mathcal{I}$ and $\mathcal{J}$ are relative left ideals then

$$\mathcal{I} -_l \mathcal{J} := \{A \in \mathbf{G} \mid A\mathcal{J} \subseteq \mathcal{I}\}.$$

Chain of ideals

Let $\mathcal{I}$ be a relative ideal of $\mathcal{S}$. We have the following chains:

$$\mathcal{S} \subseteq \mathcal{I}^* - \mathcal{I}^* \subseteq (\mathcal{I}^*)^2 - (\mathcal{I}^*)^2 \subseteq \cdots \subseteq \mathbf{G}, \tag{1}$$

and

$$\mathcal{S} \subseteq \mathcal{I}^* - (\mathtt{m}(\mathcal{I}) + \mathcal{S}) \subseteq (\mathcal{I}^*)^2 - (\mathtt{m}(\mathcal{I})^2 + \mathcal{S}) \subseteq \cdots \subseteq \mathbf{G}. \tag{2}$$

These chains were appears in the work of Barruci, Valentina, Dobbs, David, etc. for numerical semigroups.

Theorem (Can, S.)

Let $\mathcal{I}$ be a relative ideal of $\mathcal{S}$ in $\mathbf{G_P}$. Then the chains (1) and (2) stabilizes to the same unipotent numerical monoid.

Definition

We define the *blowup* of $\mathcal{I}$ to be the unipotent numerical monoid coming from these chains. It is denoted by $\mathcal{B}(\mathcal{I})$. Also,

$$\mathcal{B}(\mathcal{S}) := \mathcal{B}(\mathcal{S}^*).$$

Theorem (Can,S.)

Let $\mathcal{I}$ be a relative ideal of $\mathcal{S}$. Then we have

$$|\mathtt{m}(\mathcal{B}(\mathcal{S}))| \leq |\mathtt{m}(\mathcal{S})|$$

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Definition

A unipotent numerical monoid $\mathcal{S}$ is said to be *Arf* if for every chain

$$A \leq_{\mathbf{G},t} B \leq_{\mathbf{G},t} C$$

in $\mathcal{S}^*$, we have

$$\{CBA^{-1}, BCA^{-1}, A^{-1}CB, A^{-1}BC, CA^{-1}B, BA^{-1}C\} \subseteq \mathcal{S}.$$

Definition

The *Arf closure* of $\mathcal{S}$ is the smallest Arf monoid containing $\mathcal{S}$. We denote it by $\mathtt{Arf}(\mathcal{S})$

Theorem (Can, S.)

$$\mathtt{m}(\mathtt{Arf}(\mathcal{S})) = \mathtt{m}(\mathcal{S}).$$

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Theorem

Let $\mathcal{S}$ be a unipotent numerical monoid in **G** such that $\mathbf{1}_n \sqcup (\mathtt{Arf}(\mathcal{S})^*)^h$ is Arf for all positive integer h. Then

$$\mathcal{B}(\mathtt{Arf}(\mathcal{S})) = \mathtt{Arf}(\mathcal{B}(\mathcal{S})).$$

Future directions & problems

1. We look forward to generalizing it to linear algebraic groups. We know that $\mathbf{G} = \mathbf{R} \ltimes \mathbf{U}(n)$.
2. Derive connection to Weierstrass semigroup of multiple points on a curve X.
3. Connection to algebraic coding theory.
4. Connection to commutative algebra.

References

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THANK YOU!!!