Enumerating log rational curves on $\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^s \oplus \mathcal{O}(-a))$

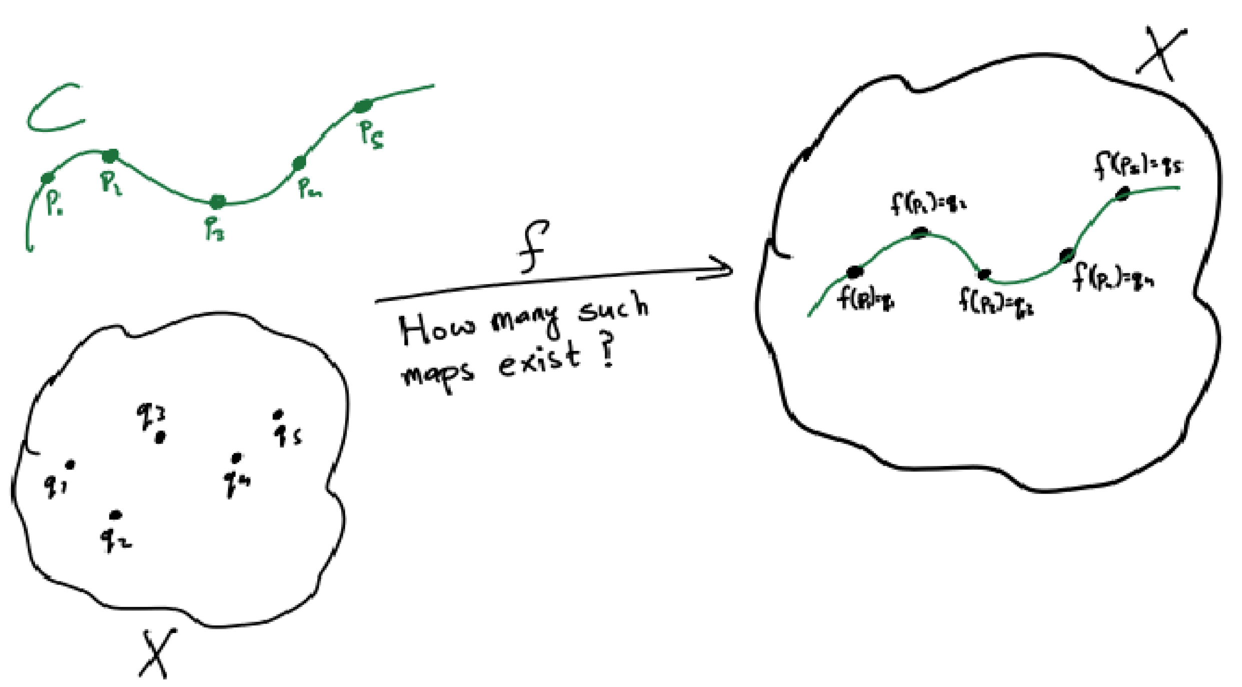
Carl Lian[1], Naufil Sakran[2]

[1]Washington University in St. Louis., MO, USA, [2]Tulane University, New Orleans, LA, USA

Motivating Question

Let X be a smooth projective variety, and let C be a general pointed curve of genus g. Fix n general points $\{p_i\}_{i=1}^n$ in C and $\{q_i\}_{i=1}^n$ in X. Let $\beta \in H_2(X,\mathbb{Z})$ be an effective curve class on X.

How many morphisms $f: C \to X$ are there in class β (i.e. $f_*([C]) = \beta$) satisfying the incidence condition $f(p_i) = q_i$ for all i?



Background

Let X be a smooth projective variety over $\mathbb{C}$, and let $\beta \in H_2(X,\mathbb{Z})$ be a non-zero effective curve class.

Fix $g, n \geq 0$ such that

$$2g - 2 + n > 0$$

so that $\mathcal{M}_{g,n}$, the moduli space of smooth n-pointed genus g curves is a Deligne–Mumford stack. Consider the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and let $\mathcal{M}_{g,n}(X,\beta)$ be the open subscheme of smooth maps. Consider the forgetful map

$$\tau : \mathcal{M}_{g,n}(X,\beta) \to \mathcal{M}_{g,n} \times X^n.$$

$$(f : (C, \{p_i\}_{i\in[n]}) \to X) \mapsto (C, \{p_i\}) \times \{f(p_i)\}_{i\in[n]}$$

Assume

$$\dim(\mathcal{M}_{g,n}(X,\beta)) = \dim(\mathcal{M}_{g,n} \times X^n).$$

Suppose all the dominating components of $\mathcal{M}_{g,n}(X,\beta)$ are generically smooth of the expected dimension.

Then the **geometric Tevelev degree** $\mathrm{Tev}^X_{g,\beta,n}$ is defined as

$$\mathrm{Tev}^X_{g,\beta,n} = \deg \tau$$

Example

Let C be a genus g curve with n general points $\{p_i\}_{i=1}^n$.

- If

$$X = \mathbb{P}^1, \quad \beta = d\cdot[H], \quad d \geq g+1, \quad n = 2d - g + 1,$$

then

$$\mathrm{Tev}^X_{g,\beta,n} = 2^g$$

- If

$$X = \mathbb{P}^r, \quad \beta = d\cdot[H], \quad d \geq r(g+1), \quad n = \frac{r+1}{r}\cdot d - g + 1,$$

then

$$\mathrm{Tev}^X_{g,\beta,n} = (r+1)^g$$

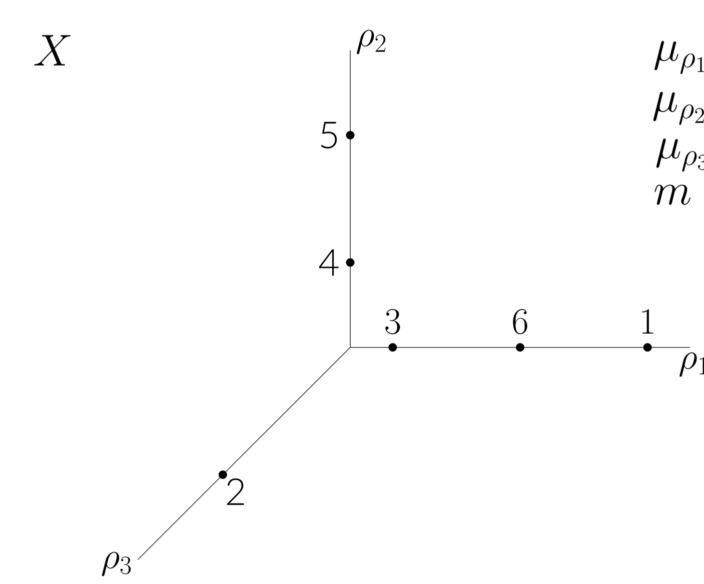
Logarithmic Tevelev Degrees with tangency conditions

Let X be a smooth, projective toric variety, and let $D_\rho \subset X$ be the torus-invariant divisors, indexed by the maximal rays $\rho \in \sum(1)$ in the fan $\sum$ of X. The **boundary** of X is given by $\bigcup_{\rho\in\sum(1)} D_\rho$ and its complement is the **interior** X° of X.

Let $\beta \in H_2(X,\mathbb{Z})$ be an effective curve class, and let $n \geq 3$. On the maximal ray ρ, suppose there are m_ρ points, and let $\mu_\rho = (\mu_{\rho,\nu})_{\nu=1}^{m_\rho} \in \mathbb{N}^{m_\rho}$ describes the multiplicities at each of the points. Let

$$m = \sum_{\rho\in(1)} m_\rho$$

denote the total number of points on the rays.



$\mu_{\rho_1} = (3,6,1)$
$\mu_{\rho_2} = (4,5)$
$\mu_{\rho_3} = (2)$
$m = 6$

Now, consider $n+m$ distinct points

$$(P,Q) = \left(\{p_i\}_{i=1}^n, \{q_{\rho,\nu\rho\in\sum(1)}\}_{\nu=1}^{m_\rho}\right) \in (\mathbb{P}^1)^{n+m}.$$

Let $\mathcal{M}_\Gamma(X)$ denote the moduli space of log maps $f : (\mathbb{P}^1, P, Q) \to X$ such that

- $f_*[\mathbb{P}^1] = \beta$,
- f maps p_i to X° and f maps $q_{\mu,\nu}$ to the boundary of X with multiplicity $\mu_{\rho,\nu}$.

Let Γ denote the data of β, the integer n, and the vectors μ_ρ specifying the tangency profile of f along the boundary. Let $n = \frac{m}{\dim X} + 1$. Consider the forgetful map

$$\tau : \mathcal{M}_\Gamma(X) \to \mathcal{M}_{0,n} \times X^n.$$

Define (genus 0) **logarithmic Tevelev degree** as

$$\mathrm{logTev}^X_\Gamma = \deg \tau.$$

Target Variety $X_{r,s,a} := \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^s \oplus \mathcal{O}(-a))$

A point $x \in X_{r,s,a}$ is given by $r+s+2$-tuple of complex numbers $(y_0, \ldots, y_{r+s+1})$ such that for $\lambda \in \mathbb{C}^*$,

$$\begin{aligned}(x_0, x_1, \ldots, x_{r+1}, x_{r+2}, \ldots, x_{r+s+1}) &\sim (\lambda^{-a}x_0, \lambda x_1, \ldots, \lambda x_{r+1}, x_{r+2}, \ldots, x_{r+s+1}) \\ &\sim (\lambda x_0, x_1, \ldots, x_{r+1}, \lambda x_{r+2}, \ldots, \lambda x_{r+s+1})\end{aligned}$$

In particular,

$$x = [x_0x_1^a : \cdots : x_0x_{r+1}^a : x_{r+2} : \cdots : x_{r+s+1}]$$

A map $f : \mathbb{P}^1 \to X_{r,s,a}$ is given by the data of $r+s+2$ sections g_i such that

- $g_1, \ldots, g_r \in H^0(\mathbb{P}^1, \mathcal{O}(b))$,
- $g_0 \in H^0(\mathbb{P}^1, \mathcal{O}(c-ab))$, and
- $g_{r+2}, \ldots, g_{r+s+1} \in H^0(\mathbb{P}^1, \mathcal{O}(c))$.

Here $c - ab \geq 0$ and the triple (a,b,c) determines the curve class β. In particular,

$$f = [g_0g_1^a : \cdots : g_0g_{r+1}^a : g_{r+2} : \cdots : g_{r+s+2}].$$

Main Theorem [2]

Consider the space $X = X_{r,s,a}$ where $r, s \geq 1$ and $a \geq 0$. Let Γ denote the tangency data for maps to X. Assume that $n = \frac{m}{r+s} + 1 \geq 3$. If the inequalities

1. $m_j \leq n-1$ for all $j = 1, \ldots, r+s+1$,
2. $\sum_{j=r+2}^{r+s+1} m_j \geq (s-1)(n-1)$, and
3. $m_0 + \sum_{j=r+2}^{r+s+1} m_j \leq s(n-1)$

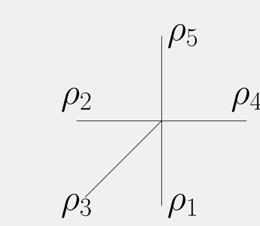
holds, then

$$\mathrm{logTev}^X_\Gamma = \left(\prod_{j=0}^{r+s+1} m_j!\right)\left(\prod_{j=0}^{r+s+1}\prod_{\nu=1}^{m_j} \mu_{j,\nu}\right) a^{\sum_{j=0}^{r+1} m_j - r(n-1) - m_0} \binom{\sum_{j=0}^{r+1} m_j - r(n-1)}{m_0}.$$

Otherwise, $\mathrm{logTev}^{X_{r,s,a}}_\Gamma = 0$. The expression 0^0 is interpreted to equal 1.

Enumerativity of $X = \mathrm{Bl}_{[0:1:0],[0:0:1]}(\mathbb{P}^2)$

A point $x \in X$ is given in coordinates as $x = [x_1x_2x_3 : x_1x_4 : x_2x_5]$. Fan of X is given by



A degree d map $f : \mathbb{P}^1 \to X$ is given by five divisors $\{D_i\}$ and five sections g_i such that

- $g_1 \in H^0(\mathbb{P}^1, \mathcal{O}(d)(-D_1))$, $g_2 \in H^0(\mathbb{P}^1, \mathcal{O}(d)(-D_2))$,
- $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}(d)(-D_1 - D_2 - D_3))$,
- $g_4 \in H^0(\mathbb{P}^1, \mathcal{O}(d)(-D_1 - D_4))$, $g_5 \in H^0(\mathbb{P}^1, \mathcal{O}(d)(-D_2 - D_5))$.

In particular, $f = [g_1g_2g_3 : g_1g_4 : g_2g_5]$.

Theorem [2]

Consider the tangency data Γ as given above. Suppose $n = \frac{m+2}{2} \geq 3$. If

$$\max\{m_1 + m_3, m_1 + m_5, m_2 + m_4\} \leq n - 1,$$

then the formula aligns with the formula conjectured in [1], that is

$$\mathrm{logTev}^X_\Gamma = \left(\prod_{j=1}^{5} m_j!\right)\left(\prod_{j=1}^{5}\prod_{\nu=1}^{m_j} \mu_{j,\nu}\right)\binom{n-1-m_5}{m_1}\binom{n-1-m_4}{m_2}.$$

The inequalities do not cover all the possible cases, and a correct formula for the remaining cases is still **unknown**. For instance, take

$$\mu_1 = \mu_2 = (1), \quad \mu_3 = (1,1,1,1), \quad \mu_4, \mu_5 = (5),$$

which gives $m = 8$, $n = 5$ and $d = 6$. Then $\mathrm{logTev}^X_\Gamma = 2400$. But if we were to substitute the values in the above formula, we would get 5400.

References

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