Unipotent Numerical Semigroups

Arf Semigroups

Naufil Sakran
Joint work with Dr. Mahir Bilen Can

Department of Mathematics
Tulane University

November 4, 2023

Table of Contents

Table of Contents

Numerical Semigroups

Definition

A subset $\mathcal{S} \subseteq \mathbb{Z}_{\geq 0}$ is a numerical semigroup if

- $0 \in \mathcal{S}$.
- If $a, b \in \mathcal{S}$ then $a + b \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbb{Z}_{\geq 0}$ is finite.

Numerical Semigroups

Definition

A subset $\mathcal{S} \subseteq \mathbb{Z}_{\geq 0}$ is a numerical semigroup if

- $0 \in \mathcal{S}$.
- If $a, b \in \mathcal{S}$ then $a + b \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathbb{Z}_{\geq 0}$ is finite.

Example

Let $\mathcal{S} = \{0, 3, 5, 6, 8, 9, 10, \rightarrow\} = \langle 3, 5 \rangle$.

Invariants

Let $\mathcal{S}$ be a numerical semigroup.

- Multiplicity $\mathtt{m}(\mathcal{S})$ is the smallest non-zero number in $\mathcal{S}$.
- Gap set $\mathtt{G}(\mathcal{S})$ is the set of elements of the complement of $\mathcal{S}$ in $\mathbb{Z}_{\geq 0}$. Genus $\mathtt{g}(\mathcal{S})$ is the cardinality of $\mathtt{G}(\mathcal{S})$.
- Frobenius element $\mathtt{F}(\mathcal{S})$ is the largest number in the gap set $\mathtt{N}(\mathcal{S})$.
- $\mathtt{PF}(\mathcal{S}) := \{x \in \mathtt{G}(\mathcal{S}) : x + \mathcal{S} \subseteq \mathcal{S}\}$
- Conductor $\mathtt{c}(\mathcal{S}) = \mathtt{F}(\mathcal{S}) + 1$.
- Sporadic elements $\mathtt{N}(\mathcal{S}) := \{x \in \mathcal{S} : x < \mathtt{F}(\mathcal{S})\}$. We denote $\mathtt{n}(\mathcal{S}) = |\mathtt{N}(\mathcal{S})|$.
- Minimal generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$.

Example

Let $\mathcal{S} = \{0, 3, 5, 6, 8, 9, 10, \to\} = \langle 3, 5 \rangle$.

- $\mathtt{m}(\mathcal{S}) = 3$.
- $\mathtt{G}(\mathcal{S}) = \{1, 2, 4, 7\}$ and $\mathtt{g}(\mathcal{S}) = 4$.
- $\mathtt{F}(\mathcal{S}) = 7$.
- $\mathtt{c}(\mathcal{S}) = 8$.
- $\mathtt{PF}(\mathcal{S}) = \{4, 7\}$
- $\mathtt{N}(\mathcal{S}) = \{0, 3, 5, 6\}$ and $\mathtt{n}(\mathcal{S}) = 4$.
- $\mathtt{e}(\mathcal{S}) = 2$

Applications

- Let $\mathcal{S}$ be a numerical semigroup. Let $\mathbb{K}$ be algebraically closed and define $\mathbb{K}[\mathcal{S}] = \oplus_{s\in\mathcal{S}}\mathbb{K}t^s$. Consider $\mathbb{K}[[\mathcal{S}]]$.
  [Kun70] showed that $\mathbb{K}[[\mathcal{S}]]$ is a Gorenstein ring if and only if $\mathtt{F}(\mathcal{S})$ is odd and $\mathcal{S}$ is irreducible.

Applications

- Let $\mathcal{S}$ be a numerical semigroup. Let $\mathbb{K}$ be algebraically closed and define $\mathbb{K}[\mathcal{S}] = \oplus_{s \in \mathcal{S}} \mathbb{K} t^s$. Consider $\mathbb{K}[[\mathcal{S}]]$.
  [Kun70] showed that $\mathbb{K}[[\mathcal{S}]]$ is a Gorenstein ring if and only if $\mathtt{F}(\mathcal{S})$ is odd and $\mathcal{S}$ is irreducible.
- Application to One-Dimensional Analytically Irreducible Local Domains (*i.e. One-Dimensional Local Rings* (A, m) *such that the m-adic completion of A is a domain.*)

Applications

- Let $\mathcal{S}$ be a numerical semigroup. Let $\mathbb{K}$ be algebraically closed and define $\mathbb{K}[\mathcal{S}] = \oplus_{s \in \mathcal{S}} \mathbb{K}t^s$. Consider $\mathbb{K}[[\mathcal{S}]]$.
  [Kun70] showed that $\mathbb{K}[[\mathcal{S}]]$ is a Gorenstein ring if and only if $\mathtt{F}(\mathcal{S})$ is odd and $\mathcal{S}$ is irreducible.
- Application to One-Dimensional Analytically Irreducible Local Domains (*i.e. One-Dimensional Local Rings* (A, m) *such that the m-adic completion of A is a domain.*)
- Let X be a compact Reimann surface or equivalently, a smooth algebraic projective curve over the complex field $\mathbb{C}$ of genus g. For each $P \in X$, there are exactly g integers $\alpha_i(P)$ with

$$1 = \alpha_1(P) < \cdots < \alpha_g(P) \leq 2g - 1$$

  such that there exist no meromorphic function on X having a pole at P, of multiplicity $\alpha_i(P)$, as its only singularity [Del08].

Table of Contents

Unipotent Numerical Semigroups

Let

$$\mathbf{U}(n, \mathbb{Z}_{\geq 0}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{ij}\}_{1<i<j<n} \in \mathbb{Z}_{\geq 0} \right\}$$

Unipotent Numerical Semigroups

Let

$$\mathbf{U}(n,\mathbb{Z}_{\geq 0}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{ij}\}_{1<i<j<n} \in \mathbb{Z}_{\geq 0} \right\}$$

We fix a finitely generated monoid $\mathcal{M} \subseteq \mathbf{U}(n,\mathbb{Z}_{\geq 0})$.

Unipotent Numerical Semigroups

Let

$$\mathbf{U}(n,\mathbb{Z}_{\geq 0}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{ij}\}_{1<i<j<n} \in \mathbb{Z}_{\geq 0} \right\}$$

We fix a finitely generated monoid $\mathcal{M} \subseteq \mathbf{U}(n,\mathbb{Z}_{\geq 0})$. A subset $\mathcal{S} \subseteq \mathcal{M}$ is a *unipotent numerical semigroup* if

- $\mathbf{1}_n \in \mathcal{S}$.
- If $A, B \in \mathcal{S}$ then $AB \in \mathcal{S}$.
- Complement of $\mathcal{S}$ in $\mathcal{M}$ is finite.

Let us fix $\mathcal{M} = \mathbf{P}(n, \mathbb{Z}_{\geq 0})$ where

$$\mathbf{P}(n, \mathbb{Z}_{\geq 0}) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \{x_{1j}\}_{1<j\leq n} \in \mathbb{Z}_{\geq 0} \right\} \quad (\cong \mathbb{Z}_{\geq 0}^{n-1})$$

Example

Let

$$\mathbf{P}_k(n) := \{(x_{1j})_{1<j\leq n} \in \mathbf{P}(n) : x_{1j} \geq k \text{ for some } 1 > j \leq n\} \subseteq \mathbf{P}(n).$$

Example

Let

$$\mathbf{P}_k(n) := \{(x_{1j})_{1<j\leq n} \in \mathbf{P}(n) : x_{1j} \geq k \text{ for some } 1 > j \leq n\} \subseteq \mathbf{P}(n).$$

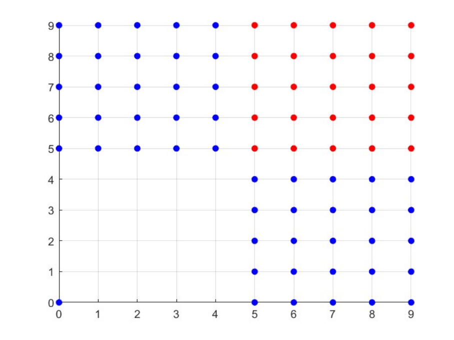


Figure: This is $\mathbf{P}_5(3)$

Example

Let $\mathcal{S} \subseteq \mathbf{P}(3)$ and consider $\mathcal{S}$ plotted as

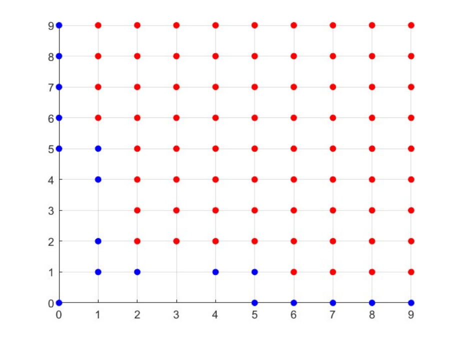


Figure: $\mathcal{S} = \langle (1,1), (2,1), (1,2), (4,1), (1,4), \mathbf{P}_5 \rangle$

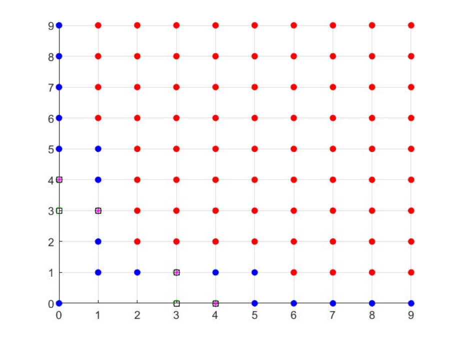
Invariants

Let $\mathcal{S} \subseteq \mathcal{M}$ be a unipotent numerical semigroup.

- Multiplicity $\mathtt{m}(\mathcal{S}) := \{(x_{ij}) \in \mathcal{S} : \sum x_{ij} \text{ is minimum}\}$
- Gap set $\mathtt{G}(\mathcal{S})$ is the set of elements of the complement of $\mathcal{S}$ in $\mathcal{M}$. Genus $\mathtt{g}(\mathcal{S}) = |\mathtt{G}(\mathcal{S})|$.
- Frobenius elements $\mathtt{F}(\mathcal{S}) := \{A \in \mathtt{N}(\mathcal{S}) : A\mathcal{M}^* \subseteq \mathcal{S}\}$.
- PseudoFrobenius elements $\mathtt{PF}(\mathcal{S}) := \{A \in \mathtt{N}(\mathcal{S}) : A\mathcal{S}^* \subseteq \mathcal{S}\}$.
- Generating number $\mathtt{r}(\mathcal{S}) = \min\{k \in \mathbb{N} : \mathbf{P}_k \subseteq \mathcal{S}\}$.
- Sporadic elements $\mathtt{N}(\mathcal{S}) := \mathcal{S} \backslash \mathcal{M}_{\mathtt{r}(\mathcal{S})}$ and $\mathtt{n}(\mathcal{S}) = |\mathtt{n}(\mathcal{S})|$.
- Minimal generating set of $\mathcal{S}$ is denoted by $\mathtt{e}(\mathcal{S})$.

Example

Let $\mathcal{S} \subseteq \mathbf{P}(3)$ and consider $\mathcal{S}$ plotted as



$\mathtt{m}(\mathcal{S}) = \{(1,1)\}, \quad \mathtt{F}(\mathcal{S}) = \{(0,4),(1,3),(3,1),(4,0)\},$
$\mathtt{r}(\mathcal{S}) = 5, \quad \mathtt{g}(\mathcal{S}) = 10,$
$\mathtt{PF}(\mathcal{S}) = \mathtt{F}(\mathcal{S}) \cup \{(0,3),(3,0)\}, \quad \mathtt{e}(\mathcal{S}) = 17.$

Table of Contents

Wilf Conjecture

Let $\mathcal{S}$ be a numerical semigroup then we always have

$$\mathtt{c}(\mathcal{S}) \leq \mathtt{e}(\mathcal{S})\mathtt{n}(\mathcal{S})$$

Wilf Conjecture

Let $\mathcal{S}$ be a numerical semigroup then we always have

$$\mathtt{c}(\mathcal{S}) \leq \mathtt{e}(\mathcal{S})\mathtt{n}(\mathcal{S})$$

Example

Let $\mathcal{S} = \{0, 5, 8, 12, \rightarrow\} = \langle 5, 8, 12, 14, 17\rangle$. We have

- $\mathtt{c}(\mathcal{S}) = 12$
- $\mathtt{e}(\mathcal{S}) = 5$
- $\mathtt{n}(\mathcal{S}) = 3$

$$12 \leq 5 * 3$$

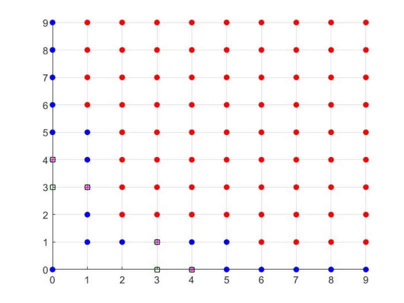
Unipotent Wilf Conjecture [CSca]

Let $\mathcal{M} \subseteq \mathbf{U}(n, \mathbb{Z}_{\geq 0})$ be a finitely generated monoid. Let $\mathcal{S}$ be a unipotent numerical semigroup in $\mathcal{M}$ then we always have

$$\mathtt{d}_{\mathcal{M}}\mathtt{c}_{\mathcal{M}}(\mathcal{S}) \leq \mathtt{e}_{\mathcal{M}}(\mathcal{S})\,\mathtt{n}_{\mathcal{M}}(\mathcal{S})$$

Example

Let $\mathcal{S} \subseteq \mathbf{P}(3)$ and consider $\mathcal{S}$ plotted as



- $\mathtt{c}(\mathcal{S}) = 25$.
- $\mathtt{d}_{\mathcal{S}} = 2$.
- $\mathtt{n}(\mathcal{S}) = 17$.
- $\mathtt{e}(\mathcal{S}) = 17$.

$$2 * 25 \leq 17 * 17$$

Table of Contents

Ideals

We are restricting ourselves to the monoid $\mathcal{M}^+ = \mathbf{P}(n, \mathbb{Z}_{\geq 0}) \subseteq \mathbf{U}(n, \mathbb{Z}_{\geq 0})$.
By abuse of notation, we write $\mathcal{M}^\pm = (-\mathcal{M}^+) \cup \mathcal{M}^+$.

Ideals

We are restricting ourselves to the monoid $\mathcal{M}^+ = \mathbf{P}(n, \mathbb{Z}_{\geq 0}) \subseteq \mathbf{U}(n, \mathbb{Z}_{\geq 0})$. By abuse of notation, we write $\mathcal{M}^\pm = (-\mathcal{M}^+) \cup \mathcal{M}^+$.

Relative ideals

Let $\mathcal{S}$ be a unipotent numerical semigroup. A subset $\mathcal{I} \subseteq \mathcal{M}^\pm$ is a relative ideal of $\mathcal{S}$ if

- $\mathcal{S}\mathcal{I} \cap \mathcal{M}^\pm \subseteq \mathcal{I}$.
- There exists $A \in \mathcal{S}$ such that $A\mathcal{I} \subseteq \mathcal{S}$.

Ideals

We are restricting ourselves to the monoid $\mathcal{M}^+ = \mathbf{P}(n, \mathbb{Z}_{\geq 0}) \subseteq \mathbf{U}(n, \mathbb{Z}_{\geq 0})$. By abuse of notation, we write $\mathcal{M}^\pm = (-\mathcal{M}^+) \cup \mathcal{M}^+$.

Relative ideals

Let $\mathcal{S}$ be a unipotent numerical semigroup. A subset $\mathcal{I} \subseteq \mathcal{M}^\pm$ is a relative ideal of $\mathcal{S}$ if

- $\mathcal{S}\mathcal{I} \cap \mathcal{M}^\pm \subseteq \mathcal{I}$.
- There exists $A \in \mathcal{S}$ such that $A\mathcal{I} \subseteq \mathcal{S}$.

Proper ideals

Let $\mathcal{S}$ be a unipotent numerical semigroup. A subset $\mathcal{I} \subseteq \mathcal{M}^\pm$ is a proper ideal of $\mathcal{S}$ if

- $\mathcal{S}\mathcal{I} \cap \mathcal{M}^\pm \subseteq \mathcal{I}$.
- There exists $A \in \mathcal{S}$ such that $A\mathcal{I} \subseteq \mathcal{S}$.
- $\mathcal{I} \subseteq \mathcal{S}$.

Example

Let $\mathcal{S}$ be a unipotent numerical semigroups.

1. $\mathcal{I} = \mathtt{PF}(\mathcal{S}) \cup \mathcal{S}$ is a relative ideal of $\mathcal{S}$.
2. $\mathcal{I} = \mathcal{S}^*$ is a proper ideal of $\mathcal{S}$.
3. Let $\mathcal{S} = \mathbf{P}_3(3)$ and consider $\mathcal{I} = \mathcal{S}^* \cup \mathbf{P}_1^*(3) \cup \{(-2, -2), (-1, -2)\}$.

Example

Let $\mathcal{S}$ be a unipotent numerical semigroups.

1. $\mathcal{I} = \mathtt{PF}(\mathcal{S}) \cup \mathcal{S}$ is a relative ideal of $\mathcal{S}$.
2. $\mathcal{I} = \mathcal{S}^*$ is a proper ideal of $\mathcal{S}$.
3. Let $\mathcal{S} = \mathbf{P}_3(3)$ and consider $\mathcal{I} = \mathcal{S}^* \cup \mathbf{P}_1^*(3) \cup \{(-2,-2),(-1,-2)\}$.

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid and $\mathcal{I}$ be a relative ideal. We define

- $\mathtt{m}(\mathcal{I}) := \{A \in \mathcal{I} \,:\, \mathsf{sum}(A) \leq \mathsf{sum}(B) \text{ for all } B \in \mathcal{I}\}$.

Example

Let $\mathcal{S}$ be a unipotent numerical semigroups.

1. $\mathcal{I} = \mathtt{PF}(\mathcal{S}) \cup \mathcal{S}$ is a relative ideal of $\mathcal{S}$.
2. $\mathcal{I} = \mathcal{S}^*$ is a proper ideal of $\mathcal{S}$.
3. Let $\mathcal{S} = \mathbf{P}_3(3)$ and consider $\mathcal{I} = \mathcal{S}^* \cup \mathbf{P}_1^*(3) \cup \{(-2,-2),(-1,-2)\}$.

Definition

Let $\mathcal{S}$ be a unipotent numerical monoid and $\mathcal{I}$ be a relative ideal. We define

- $\mathtt{m}(\mathcal{I}) := \{A \in \mathcal{I} : \mathsf{sum}(A) \leq \mathsf{sum}(B) \text{ for all } B \in \mathcal{I}\}$.
- $\mathtt{g}(\mathcal{I}) := |\, \mathtt{m}(\mathcal{I})\mathcal{M}^+ \setminus \mathcal{I} \,|$.
- $\mathtt{F}(\mathcal{I}) := \{A \in \mathcal{M}^{\pm} \setminus \mathcal{I}^* : A(\mathcal{M}^+)^* \subseteq \mathcal{I}\}$.
- $\mathtt{PF}(\mathcal{I}) := (\mathcal{I} - \mathcal{S}^*) \setminus \mathcal{I}$ and type of $\mathcal{I}$ is $\mathtt{t}(\mathcal{I}) = |\mathtt{PF}(\mathcal{I})|$.

Operation on Ideals

Recall $\mathcal{M}^{\pm} = (-\mathcal{M}^{+}) \cup \mathcal{M}^{+}$.

Example

Let $\mathcal{I}_1, \mathcal{I}_2$ be relative ideal of the unipotent numerical semigroup $\mathcal{S}$.

1. $\mathcal{I}_1 + \mathcal{I}_2 = \{AB : A \in \mathcal{I}_1,\ \&\ B \in \mathcal{I}_2\} \cap \mathcal{M}^{\pm}$.
2. $\mathcal{I}_1 - \mathcal{I}_2 := \{\mathcal{A} \in \mathcal{M}^{\pm} : A\mathcal{I}_2 \subseteq \mathcal{I}_1\}$.
3. $\mathcal{I}^k = \mathcal{I} + \cdots + \mathcal{I} \qquad \text{(k-times)}$

Table of Contents

Blowups

Definition (Reduction of $\mathcal{I}$)

Let $\mathcal{S}$ be a unipotent numerical semigroup and $\mathcal{I}$ is a relative ideal of $\mathcal{S}$ contained in $\mathcal{M}^+$ then the *reduction number* $\mathtt{r}(\mathcal{I})$ of $\mathcal{I}$ is defined as

$$\mathtt{r}(\mathcal{I}) := \min\{h \in \mathbb{N} : \mathcal{I}^{h+1} = (\mathtt{m}(\mathcal{I}) + \mathcal{S}) + \mathcal{I}^h\}$$

If $\mathtt{r}(\mathcal{I}) = 1$ then $\mathcal{I}$ is *stable*.

Blowups

Definition (Reduction of $\mathcal{I}$)

Let $\mathcal{S}$ be a unipotent numerical semigroup and $\mathcal{I}$ is a relative ideal of $\mathcal{S}$ contained in $\mathcal{M}^+$ then the *reduction number* $\mathtt{r}(\mathcal{I})$ of $\mathcal{I}$ is defined as

$$\mathtt{r}(\mathcal{I}) := \min\{h \in \mathbb{N} : \mathcal{I}^{h+1} = (\mathtt{m}(\mathcal{I}) + \mathcal{S}) + \mathcal{I}^h\}$$

If $\mathtt{r}(\mathcal{I}) = 1$ then $\mathcal{I}$ is *stable*.

Definition (Blowup)

Let $\mathcal{S}$ be a unipotent numerical semigroup with $|\mathtt{m}(\mathcal{S})| = 1$ and $\mathcal{I}$ relative ideal to $\mathcal{S}$ satisfying $\mathcal{I} \subseteq \mathcal{M}^+$ and $|\mathtt{m}(\mathcal{I})| = 1$.

Blowups

Definition (Reduction of $\mathcal{I}$)

Let $\mathcal{S}$ be a unipotent numerical semigroup and $\mathcal{I}$ is a relative ideal of $\mathcal{S}$ contained in $\mathcal{M}^+$ then the *reduction number* $\mathtt{r}(\mathcal{I})$ of $\mathcal{I}$ is defined as

$$\mathtt{r}(\mathcal{I}) := \min\{h \in \mathbb{N} : \mathcal{I}^{h+1} = (\mathtt{m}(\mathcal{I}) + \mathcal{S}) + \mathcal{I}^h\}$$

If $\mathtt{r}(\mathcal{I}) = 1$ then $\mathcal{I}$ is *stable*.

Definition (Blowup)

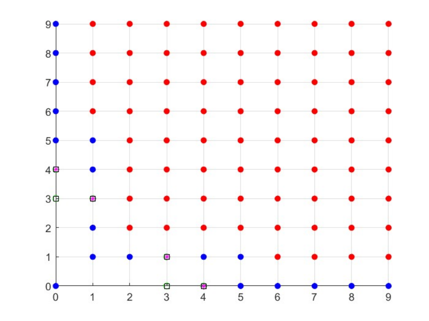
Let $\mathcal{S}$ be a unipotent numerical semigroup with $|\mathtt{m}(\mathcal{S})| = 1$ and $\mathcal{I}$ relative ideal to $\mathcal{S}$ satisfying $\mathcal{I} \subseteq \mathcal{M}^+$ and $|\mathtt{m}(\mathcal{I})| = 1$.
Then we define blowup of $\mathcal{I}$ to be

$$\mathtt{B}(\mathcal{I}) := \mathcal{I}^{\mathtt{r}(\mathcal{I})} - (\mathtt{m}(\mathcal{I})^{\mathtt{r}(\mathcal{I})} + \mathcal{S}) = \mathcal{I}^{\mathtt{r}(\mathcal{I})} - \mathcal{I}^{\mathtt{r}(\mathcal{I})}$$

The *blowup* of $\mathcal{S}$ is defined to be the blowup of $\mathcal{S}^*$ i.e $\mathtt{B}(\mathcal{S}) := \mathtt{B}(\mathcal{S}^*)$.

Example

Let $\mathcal{S} \subseteq \mathbf{P}(3)$ and consider $\mathcal{S}$ plotted as



Consider the ideal $\mathcal{S}^*$. One can see that $\mathtt{r}(\mathcal{S}^*) = 1$ as $(\mathcal{S}^*)^2 = (\mathtt{m}(\mathcal{I}) + \mathcal{S}) + \mathcal{S}^*$. So,

$$\mathtt{B}(\mathcal{S}) = \mathcal{S}^* - \mathcal{S}^* = \mathcal{S} \cup \mathtt{PF}(\mathcal{S}) = \mathcal{S} \cup \{(3,0),(4,0),(3,1),(1,3),(0,3),(0,4)\}$$

One can form a chain

$$\mathcal{S} = \mathcal{S}^0 \subseteq \mathcal{S}^1 \subseteq \mathcal{S}^2 \subseteq \cdots \subseteq \mathcal{S}^{q-1} \subseteq \mathcal{S}^q = \mathcal{M}_{\mathbb{Z}_{\geq 0}}$$

where $\mathtt{B}(\mathcal{S}^i) = \mathcal{S}^{i+1}$.

Definition (Multiplicity Sequence)

Let $\mathcal{S}$ be a unipotent numerical semigroup. Consider the chain

$$\mathcal{S} = \mathcal{S}^0 \subseteq \mathcal{S}^1 \subseteq \cdots \subseteq \mathcal{S}^q = \mathcal{M}^+$$

One can form a sequence $(\mathtt{d}_0, \mathtt{d}_1, \ldots, \mathtt{d}_q)$ where $\mathtt{d}_i$ is the sum of elements of $\mathtt{m}(\mathcal{S}^i)$. This is clearly a non-increasing sequence that ends at 1.

Example

From the previous example, the multiplicity sequence of $\mathcal{S}$ is (2,2,1).

Definition (Integrally closed)

Let $\mathcal{S}$ be a unipotent numerical semigroup and let $\mathcal{I}$ be an ideal of $\mathcal{S}$. Then $\mathcal{I}$ is said to be *intergrally closed* if

$$\mathcal{I} = \mathcal{S}(A) := (A\mathcal{M}^{+}) \cap \mathcal{S}.$$

for some $A \in \mathcal{S}$.

Definition (Arf semigroups)

A numerical semigroup $\mathcal{S}$ is said to be *Arf* if every integrally closed ideal of $\mathcal{S}$ is stable.

Definition (Integrally closed)

Let $\mathcal{S}$ be a unipotent numerical semigroup and let $\mathcal{I}$ be an ideal of $\mathcal{S}$. Then $\mathcal{I}$ is said to be *intergrally closed* if

$$\mathcal{I} = \mathcal{S}(A) := (A\mathcal{M}^{+}) \cap \mathcal{S}.$$

for some $A \in \mathcal{S}$.

Definition (Arf semigroups)

A numerical semigroup $\mathcal{S}$ is said to be *Arf* if every integrally closed ideal of $\mathcal{S}$ is stable.

Definition (Arf closure)

Let $\mathcal{S}$ be a unipotent numerical semigroup. We define the *Arf closure* of $\mathcal{S}$, to be the smallest unipotent Arf semigroup containing $\mathcal{S}$. We denote it by $\texttt{Arf}(\mathcal{S})$.

Lemma (Blowup of Arf is Arf.)

Let $\mathcal{S}$ be an Arf semigroup such that $|\mathtt{m}(\mathcal{S})| = 1$ then $\mathtt{B}(\mathcal{S})$ is also Arf.

Lemma (Multiplcity of $\mathcal{S}$ and $\mathtt{Arf}(\mathcal{S})$ coincides.)

Any numerical semigroup $\mathcal{S}$ has the same multiplicity sum as its Arf closure i.e.

$$\{\sum x_{ij} : (x_{ij}) \in \mathtt{m}(\mathcal{S})\} = \{\sum y_{ij} : (y_{ij}) \in \mathtt{m}(\mathtt{Arf}(\mathcal{S}))\}$$

Lemma

If $\mathcal{S}$ is an Arf semigroup with $|\mathtt{m}(\mathcal{S})| = 1$ and take $A \in \mathcal{S}^*$. Then $\mathcal{S}^\circ = \mathbf{1}_n \cup A\mathcal{S}$ is an Arf semigroup, whose blowup is $\mathcal{S}$.

Main Result

Theorem!!!

Blowup and Arf closure commute when $|\mathtt{m}(\mathcal{S})| = 1$. More precisely, if $\mathcal{S}$ is a unipotent numerical semigroup such that $|\mathtt{m}(\mathcal{S})| = 1$, then

$$\mathtt{Arf}(\mathtt{B}(\mathcal{S}^*)) = \mathtt{B}(\mathtt{Arf}(\mathcal{S})^*)$$

References

[Kun70] Ernst Kunz. "The value-semigroup of a one-dimensional Gorenstein ring". In: *Proceedings of the American Mathematical Society* 25.4 (1970), pp. 748–751.

[Del08] Andrea Del Centina. "Weierstrass points and their impact in the study of algebraic curves: a historical account from the "Lückensatz" to the 1970s". In: *ANNALI DELL'UNIVERSITA'DI FERRARA* 54 (2008), pp. 37–59.

[CSca] Mahir Bilen Can and Naufil Sakran. "On Generalized Wilf Conjectures". In: *arXiv preprint arXiv:2306.05530* (2023 To appear in Portugaliae Mathematica).