Unipotent Wilf Conjecture

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Wilf Conjecture

Let S be a complement finite submonoid of $\mathbb{N}_0$, (a.k.a numerical semigroup).

- The conductor of S, denoted by $c(S)$ is the smallest integer c such that $c + \mathbb{N} \subseteq S$.
- The sporadic elements of S, are elements in S that are less than c. We denote their cardinality by $n(S)$.
- The embedding dimension, $e(S)$ of S, is the cardinality of the minimal generating set of $S \backslash \{0\}$.

In 1978, Wilf conjectured that [1] for any numerical semigroup S, we have

$$c(S) \le e(S)n(S)$$

Previous Generalization

Let S be a complement finite submonoid of $\mathbb{N}_0^d$ (a.k.a generalized numerical semigroup). Let $\le$ be a partial order on $\mathbb{N}_0^d$ such that for $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbb{N}_0^d$, $x \le y$ if and only if $x_i \le y_i$ for all $i = 1, \dots, r$. Let $H(S) = \mathbb{N}_0^d \backslash S$. We define

- The conductor of S, denoted by $\mathrm{c}(S)$ is the cardinality of the set
$$\{x \in \mathbb{N}_0^d \, : \, x \le h \text{ for some } h \in H(S)\}$$
- Let $\mathrm{n}(S)$ denote the cardinlity of the set
$$\{x \in S \, : \, x \le h \text{ for some } h \in H(S)\}$$
- Let $\mathrm{e}(S)$ denote the cardinality of the minimal set of generators of S.

Generalized Wilf Conjecture [2] states that

$$d\mathrm{c}(S) \le \mathrm{e}(S)\mathrm{n}(S)$$

Notations

Let G be a unipotent complex linear algebraic group and let $M = G_{\mathbb{N}}$. Let $S \subseteq M$ be complement finite submonoid. We define

- The generating number of S is defined as $\mathbf{r}_M(S) = \min\{k \in \mathbb{N} : \mathbf{U}(n, \mathbb{N}_0)_{\mathbf{r}_M(S)} \subseteq S\}$.
- $\mathbf{d}_M := \dim G$.
- $\mathbf{c}_M(S) := \mathrm{r}(S)^{\mathrm{d}_M}$. *(Conductor of S.)*
- $\mathbf{n}_M(S) := |S \backslash \mathbf{U}(n, \mathbb{N})_{\mathbf{r}_M(S)}| + 1$.
- $\mathbf{e}(S) := \min\{|\mathscr{G}| \, : \, \mathscr{G} \text{ generates } S \backslash \{1_n\}\}$.
- $\mathbf{g}(S) := |M \backslash S|$. *(Genus of S relative to M.)*

Important families

Let $G = \mathbf{U}(n, \mathbb{C})$ be the group unipotent upper triangular $n \times n$ matrices with entries in $\mathbb{C}$. Define $M \subseteq G_{\mathbb{N}}$ as

$$\mathbf{P}(n, \mathbb{N}_0) := \left\{ \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : a_i \in \mathbb{N}_0 \right\}$$

The previous generalization is a special case of the above M. Define $\mathbf{P}(n, \mathbb{N}_0)_k = \{(x_{ij}) : k \le \max_{1 \le i < j \le n}\{x_{ij}\}\}$. If $S \subseteq M$ be a complement finite submonoid then $\mathbf{P}(n, \mathbb{N}_0)_k \subseteq S$ for some large k.

Unipotent Wilf Conjecture!!!

Let G be an unipotent algebraic group. If S be a complement finite submonoid of the arithmetic submonoid $M = G_{\mathbb{N}}$, then we have

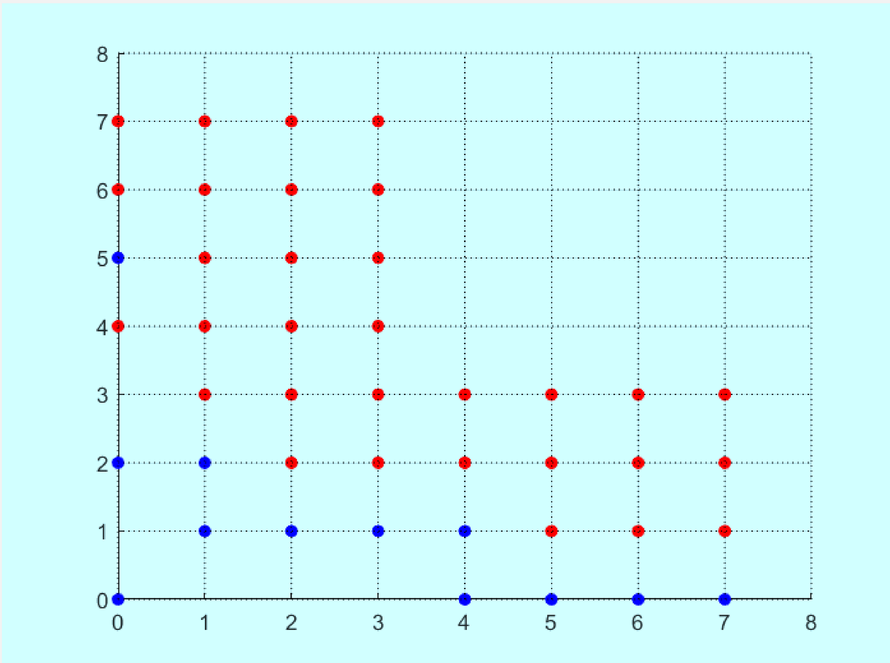
$$\mathbf{d}_M \mathbf{c}_M(S) \le \mathbf{e}(S)\mathbf{n}_M(S).$$

Thick families in $\mathbf{P}(n, \mathbb{N}_0)$

Let $S \subseteq M$ be complement finite submonoid. Let

$$S_j = S \cap (\{0\} \times \cdots \times \mathbb{N}_0 \times \cdots \times \{0\})$$

Define n_j, c_j and g_j of S_j accordingly. If $\sum_{j=1}^{n-1} g_j = \mathbf{g}_M(S)$ then S is called **thick** submonoid of M. For example

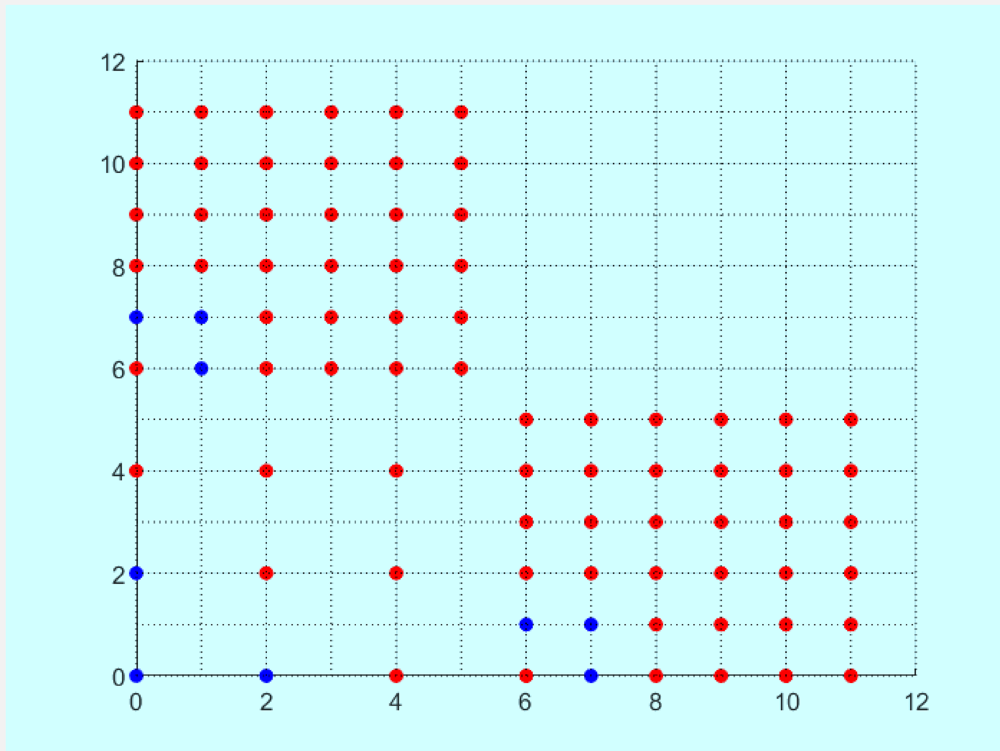


UWC holds for thick family.

Thin families in $\mathbf{P}(n, \mathbb{N}_0)$

If $\prod_{j=1}^{n-1} n_j = \mathbf{n}_M(S)$ then S is called **thin** submonoid of M.

Define $S = \{\mathbf{1}_2, (2,1), (1,3), (3,2), \mathbf{P}_5\} \subseteq M$.



We have $\mathbf{e}(S) = 23$, $\mathbf{n}_M(S) = 6$ and $\mathbf{c}_M(S) = 50$.

If S is thin and $\prod_{j=1}^{n-1} c_j = k^{n-1}$ then **UWC** holds.

Connection with Algebraic Geometry

Let $X = \{[x; y; z] \in \mathbb{P}^2 \, : \, x^3 - y^2 * z = 0\}$ be a smooth projective variety of genus 1. Putting $z = 1$, we get the affine veriety $Y = V(x^3 - y^2)$. Let $P = (1,1)$, $Q = (1,-1) \in Y$. For any $f = \frac{g}{h} \in k(X)$, $(f)_\infty = \mathrm{ord}_P(h)$, where

$$\mathrm{ord}_P(h) := \max\{k \, : \, h \in \mathfrak{m}_P^k, \, h \notin \mathfrak{m}_P^{k+1}\}$$

For the point P and Q, let $\mathfrak{m}_P = (x - y)$ and $\mathfrak{m}_Q = (x + y)$ be the maximal ideal of the localization at P and Q respectively.

As $(\frac{x+y}{(x-y)(x+y)})_\infty = (\frac{x+y}{x^2+x^3})_\infty =$, so there are no positive integer n for which $(f)_\infty \ne n$. With Macaulay2, one can see that genus of the curve is 0.

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