Ramanujan Congruences

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1. Modular Forms, Eisenstein Series & Partitions
2. Derivatives and Serre Derivatives
3. The Ramanujan Congruences

Modular Forms

Definition (Modular Forms)

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k if $f(z)$ is uniformly bounded in the region $\mathrm{Im}(z) > 1$ and for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{SL}_2(\mathbb{Z})$, we have

$$f(\gamma z) := f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

One can observe that

1. As $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have $f(\gamma z) = f\left(\frac{z+1}{0z+1}\right) \implies f(z+1) = f(z)$.
2. As $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have $f(\gamma z) = f\left(\frac{0z-1}{z+0}\right) \implies f(-\frac{1}{z}) = z^k f(z)$.

(To show f is a modular form, it suffices to show f satisfies these conditions.)

Definition

For $k \in \mathbb{Z}$, define $M_k(\mathrm{SL}_2(\mathbb{Z}))$ to be the set of all modular forms of weight k over $SL_2(\mathbb{Z})$.

Theorem

For $z \in \mathbb{H}$ and k an integer greater than 2, the series

$$G_k(z) := \sum_{\substack{m,n \in \mathbb{Z}, \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}$$

defines a modular form of weight k for all even k. For odd $k > 2$, we have $G_k(z) = 0$.

Theorem

For any even $k \geq 4$,

$$G_k(z) = 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}$$

where $\sigma_s(n) = \sum_{d|n} d^s$.

Definition (Bernoulli numbers)

We define the Bernoulli numbers B_k by

$$\frac{x}{e^x - 1} := \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}$$

which is a holomorphic function in a region around 0. e.g. $B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \ldots$

Definition (Eisenstein series)

For all even $k > 2$, we define the Eisenstein series as

$$E_k(z) := \frac{G_k(z)}{2\zeta(z)} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}$$

Some properties.

1. $G_2(z+1) = G_2(z)$ and $G_2(-\frac{1}{z}) = z^2 G_2(z) - 2\pi i z$
2. $E_2(z+1) = E_2(z)$ and $E_2(-\frac{1}{z}) = z^2 E_2(z) - \frac{6}{i\pi} z$
3. If $4 \leq k \leq 2\mathbb{Z}$, then $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.
4. $M_6(\mathrm{SL}_2(\mathbb{Z}))$ and $M_8(\mathrm{SL}_2(\mathbb{Z}))$ are vector spaces spanned by E_6 and E_8 respectively.
5. $E_4^2 = E_8,\ E_4 E_6 = E_{10},\ E_4 E_{10} = E_{14},\ E_6 E_8 = E_{14}$

Partitions

Definition

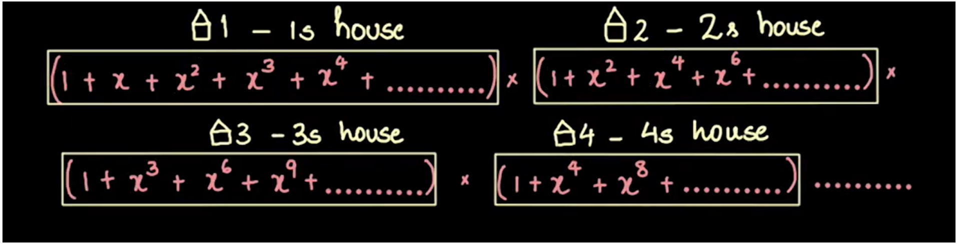
A partition of a nonnegative integer n is a non-increasing sequence of positive integers that sum to n. It is denoted as $p(n)$.

Lemma

The partition function $p(n)$ is the number of ways to represent $n \in \mathbb{N} \cup \{0\}$ as the sum of at most n positive integers. One can represent it in terms of generating function as

$$\sum_{n=0}^{\infty} p(n)x = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\cdots$$



Numbers of ways for which we can write n = Coefficient of x^n in the expansion.

Defining

$$\Delta(z) := \frac{1}{1728}(E_4^3 - E_6^2)$$

we have

Theorem

For $q = e^{2\pi iz}$

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q\left(\sum_{n=0}^{\infty} p(n)q^n\right)^{-24}$$

Derivatives

Definition

We define D as the differential operator such that

$$Df(z) = q\frac{d}{dq}f = \frac{1}{2\pi i}\frac{d}{dz}f$$

For $f \in M_k(\mathrm{Sl}_2(\mathbb{Z}))$, we define the Serre derivative θ_k as

$$\theta_k(f) = Df - \frac{k}{12}E_2 f$$

where $q = e^{2\pi i z}$

Theorem

We have that θ_k is a linear operator and satisfies the product rule for modular forms i.e. $f \in M_k$, $g \in M_{k'}$,

$$\theta_{k+k'}(fg) = g\theta_k(f) + f\theta_{k'}(g)$$

Proof. Clearly $fg \in M_{k+k'}$.

$$\begin{aligned}
\theta_{k+k'}(fg) &= \frac{1}{2\pi i}\frac{d}{dz}(fg) - \frac{k+k'}{12}E_2(z)\cdot fg \\
&= \frac{1}{2\pi i}\left(f\frac{d}{dz}g + g\frac{d}{dz}f\right) - \frac{k}{12}g\,E_2(z)\cdot f - \frac{k'}{12}f\,E_2(z)\cdot g \\
&= g\,\theta_k(f) + f\,\theta_{k'}(g).
\end{aligned}$$

Theorem

$$\theta_k : M_k(Sl_2(\mathbb{Z})) \to M_{k+2}(Sl_2(\mathbb{Z}))$$

Proof. Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

- $$\frac{d}{dz} f(\gamma z) = \frac{d}{dz}\left(f\left(\frac{az+b}{cz+d}\right)\right) = f'(\gamma z) \cdot \frac{1}{(cz+d)^2}$$
- Also
$$\frac{d}{dz} f(\gamma z) = \frac{d}{dz}((cz+d)^k f(z)) = f'(z)(cz+d)^k + ckf(z)(cz+d)^{k-1}$$

Hence,

$$f'(\gamma z) = f'(z)(cz+d)^{k+2} + ckf(z)(cz+d)^{k+1}$$

- So,

$$\theta_k f(\gamma z) = \frac{1}{2\pi i}\left(f'(z)(cz+d)^{k+2} + ckf(z)(cz+d)^{k+1}\right) - \frac{k}{12}E_2(\gamma z)f(z)(cz+d)^k$$

- Recall $E_2(z+1) = E_2(z)$ and $E_2(-\frac{1}{z}) = z^2 E_2(z) + \frac{6z}{i\pi}$. So,

$$\theta_k f(z+1) = Df(z+1) - \frac{k}{12}E_2(z+1)f(z+1) = Df(z) - \frac{k}{12}E_2(z)f(z) = \theta_k f(z)$$

and

$$\begin{aligned}\theta_k f\left(-\frac{1}{z}\right) &= \frac{1}{2\pi i}\left(f'(z)z^{k+2} + kf(z)z^{k+1}\right) - \frac{k}{12}E_2(z)f(z)z^{k+2} - \frac{k}{2\pi i}f(z)z^{k+1} \\ &= \frac{1}{2\pi i}f'(z)z^{k+2} - \frac{k}{12}E_2(z)f(z)z^{k+2} \\ &= \theta_k f(z)\cdot z^{k+2}. \quad \blacksquare\end{aligned}$$

Definition

We define the functions

$$P = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

$$Q = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

$$R = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$$

where $\sigma_s(n) = \sum_{d|n} d^s$. Here $q = e^{2\pi iz}$. *These are just E_2, E_4 and E_6 respectively.*

Theorem (The Ramanujan Derivative Identities)

We have

$$DP = \frac{1}{12}(P^2 - Q)$$
$$DQ = \frac{1}{3}(PQ - R)$$
$$DR = \frac{1}{2}(PR - Q^2)$$

Proof. Recall that $M_6(\mathrm{SL}_2(\mathbb{Z}))$ and $M_8(\mathrm{SL}_2(\mathbb{Z}))$ are 1-dimensional vector spaces spanned by E_6 and E_8 respectively and $\theta_4 E_4 \in M_6(\mathrm{SL}_2(\mathbb{Z}))$ and $\theta_6 E_6 \in M_8(\mathrm{SL}_2(\mathbb{Z}))$.

- For some $c_1, c_2 \in \mathbb{C}$

$$\theta_4(Q) = DQ - \frac{4}{12}E_2 Q = c_1 E_6 = c_1 R \quad \text{and} \quad \theta_6(R) = DR - \frac{6}{12}E_2 R = c_2 E_8 = c_2 Q^2$$

- $$DQ - \frac{1}{3}E_2 Q = 240\sum_{n=1}^{\infty} \sigma_3(n) n q^n - \frac{1}{3}(PQ)$$

  and

$$DR - \frac{1}{2}E_2 R = -504\sum_{n=1}^{\infty} \sigma_5(n) n q^n - \frac{1}{2}(PR)$$

- Comparing the coefficient of q^0, one gets $c_1 = -\frac{1}{3}$ and $c_2 = -\frac{1}{2}$.

- Hence,

$$DQ = \frac{1}{3}(PQ - R) \quad and \quad DR = \frac{1}{2}(PR - Q^2)$$

- For the last identity, we first show $\theta_2 E_2 \in M_4(\mathrm{SL}_2(\mathbb{Z}))$.

$$\theta_2 E_2(z+1) = \frac{1}{2\pi i} E_2'(z+1) - \frac{1}{12} E_2^2(z+1) = \frac{1}{2\pi i} E_2'(z) - \frac{1}{12} E_2^2(z) = \theta_2 E_2(z)$$

and

$$\begin{aligned}\theta_2 E_2(z+1) &= \frac{z^2}{2\pi i}\frac{d}{dz} E_2\left(-\frac{1}{z}\right) - \frac{1}{12}\left(z^2 E_2(z) + \frac{6}{\pi i} z\right)^2 \\ &= \frac{z^2}{2\pi i}\frac{d}{dz}\left(z^2 E_2(z) + \frac{6}{\pi i} z\right) - \frac{1}{12}\left(z^2 E_2(z) + \frac{6}{\pi i} z\right)^2 \\ &= z^4 \theta_2 E_2(z)\end{aligned}$$

- $\theta_2 E_2(z) = DP - \frac{1}{12}P^2 = c_3 Q$.

$$DP - \frac{1}{12}P^2 = -24\sum_{n=1}^{\infty}\sigma_1(n)nq^n - \frac{1}{12}\left(P^2\right)$$

  Comparing the coefficient of q^0, one gets $c_3 = -\frac{1}{12}$.

- Hence,

$$DP = \frac{1}{12}(P^2 - Q)$$

Theorem (Ramanujan)

For all integers $n \geq 0$, we have

$$p(5n+4) \equiv 0 \mod 5$$

Proof. First, note that

- $n^5 \equiv n \mod 5$. This implies

$$\sigma_5(n) = \sum_{d|n} d^5 \equiv \sum_{d|n} d = \sigma_1(n) \mod 5$$

- $Q \equiv 1 \mod 5$.
- $P \equiv R \mod 5$.

- $1728\Delta \equiv 3\Delta \equiv Q^3 - R^2 \equiv Q - P^2 \mod 5.$
- $12DP \equiv 3DP \equiv Q - P^2 \mod 5.$
- $DP \equiv \Delta \mod 5.$
- Recall that

$$\Delta \equiv q\prod_{n=1}^{\infty}(1-q^n)^{24} \qquad \text{and} \qquad \prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n$$

- $$\Delta \equiv q\prod_{n=1}^{\infty}(1-q^n)^{24} \equiv q\prod_{n=1}^{\infty}\frac{(1-q^n)^{25}}{1-q^n} \equiv q\prod_{n=1}^{\infty}(1-q^n)^{-1}\prod_{n=1}^{\infty}(1-q^{25n}) \mod 5$$

- But

$$q\sum_{n=0}^{\infty} p(n)q^n = \sum_{n=1}^{\infty} p(n-1)q^n \mod 5$$

- Also as

$$DP = q\frac{d}{dq}P = q\frac{d}{dq}(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n) = 24q\sum_{n=1}^{\infty} n\sigma_1(n)q^{n-1} = 24\sum_{n=1}^{\infty} n\sigma_1(n)q^n$$

- So,

$$24\sum_{n=1}^{\infty} n\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} p(n-1)q^n \prod_{n=1}^{\infty}(1-q^{25n}) \mod 5$$

- Note that whenever $5|n$, the L.H.S is 0. Now comparing the coefficient of q^{5k} for $k \in \mathbb{N}$, we get

$$p(5k-1) \equiv 0 \mod 5$$

and hence the result [2].■

Theorem (Ramanujan)

For all integers $n \geq 0$, we have

$$p(7n-2) \equiv 0 \mod 7$$

Proof. First, note that

- As $7|504$, $R \equiv 1 \mod 7$ and $DR \equiv 0 \mod 7$.
- As $DR = \frac{1}{2}(PR - Q^2)$ so,
$$Q^2 \equiv P \mod 7$$
- $$(Q^3 - R^2)^2 \equiv (PQ-1)^2 \equiv PQ(PQ-1) - (PQ-R) = P(P^2-Q) - (PQ-R) \mod 7$$

- $P^2 - Q = 12DP$ and $PQ - R = 3DQ$.
- $$(Q^3 - R^2)^2 \equiv 12P\, DP - 3DQ \mod 7$$
- Note that $12P\, DP = 6(2P)q\frac{dP}{dq} \equiv -q\frac{d(P^2)}{dq} \mod 7$
- $$(Q^3 - R^2)^2 \equiv -q\frac{d(P^2)}{dq} - 3q\frac{dQ}{dq} = -D(P^2 + 3Q) \mod 7$$
- Note that $P^2 + 3Q = 4 + \sum_{n=1}^{\infty} a(n)q^n$ where $a(n)$ is some sum of divisor function σ_s depending on n.

$$-D(P^2 + 3Q) = -\sum_{n=1}^{\infty} na(n)q^n$$

- $$(Q^3 - R^2)^2 = (1728\Delta)^2 \equiv \Delta^2 \equiv (q\prod_{n=1}^{\infty}(1-q^n)^{24})^2 = q^2\prod_{n=1}^{\infty}(1-q^n)^{48} \mod 7$$
- $$-D(P^2+3Q) \equiv q^2\prod_{n=1}^{\infty}\frac{(1-q^n)^{49}}{(1-q^n)} \equiv q^2\sum_{n=0}^{\infty}p(n)q^n\left(\prod_{n=1}^{\infty}(1-q^n)^{49}\right) \mod 7$$
- $$-\sum_{n=1}^{\infty}na(n)q^n \equiv \sum_{n=2}^{\infty}p(n-2)q^n\left(\prod_{n=1}^{\infty}(1-q^{49n})\right) \mod 7$$

- Note that whenever $7|n$, the L.H.S is 0. Now comparing the coefficient of q^{7k} for $k \in \mathbb{N}$, we get

$$p(7k-2) \equiv 0 \mod 7$$

and hence the result [1]. ■

Theorem (Ramanujan)

For all integers $n \geq 0$, we have

$$p(11n+6) \equiv 0 \mod 11$$

Theorem (Ahlgren and Boylan)

Let ℓ be a prime. Then

$$p(\ell n + c) \equiv 0 \mod \ell$$

for all $n \in \mathbb{N}$ iff $(\ell, c) \in \{(5,4), (7,5), (11,6)\}$

Theorem (Watson (1938), Atkin (1967))

For prime $\ell \geq 5$ and positive integer r, define $0 \leq c_{\ell,r} < \ell^r$ such that $24c_{\ell,r} \equiv 1 \mod \ell^r$. Then for every nonnegative integer n, we have

$$p(5^r n + c_{5,r}) \equiv 0 \mod 5^r$$
$$p(7^r n + c_{7,r}) \equiv 0 \mod 7^{\lfloor r/2 \rfloor + 1}$$
$$p(11^r n + c_{11,r}) \equiv 0 \mod 11^r$$

Theorem (Ahlgren, Ono (2000))

For every modulus L coprime to 6, there exists $A \neq 0$ and B such that for all $n \in \mathbb{N}$, we have

$$p(An + B) \equiv 0 \mod L$$

Example

For all $n \in \mathbb{N}$, we have

$$p(4063467631n + 30064597) \equiv 0 \mod 31$$

Fractional partition function

Definition (Chan-Wang (2018))

The fractional partition function $p_\alpha(n)$ are defined for $\alpha \in \mathbb{Q}$ by

$$\sum_{n=0}^{\infty} p_\alpha(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^\alpha$$

(Note that for $\alpha = -1$, we have our usual partition function)

Theorem (Chan-Wang (2018))

For all $n \in \mathbb{N}$, if $24c \equiv -\alpha \mod \ell$ and any of the following conditions hold:

1. $\alpha \equiv 4, 8, 14 \mod \ell$ *and* $\ell \equiv 5 \mod 6$.
2. $\alpha \equiv 6, 10 \mod \ell$ *and* $\ell \equiv 3 \mod 4$ *and* $\ell \geq 5$.
3. $\alpha \equiv 26 \mod \ell$ *and* $\ell \equiv 11 \mod 12$.

Then we have

$$p_\alpha(\ell n + c) \equiv 0 \mod \ell$$

Example

1. $p_{-\frac{3}{4}}(43n + 39) \equiv 0 \mod 43$
2. $p_{\frac{1}{3}}(41n + 37) \equiv 0 \mod 41$

[1] Bruce C Berndt. "Ramanujan's congruences for the partition function modulo 5, 7, and 11". In: *International Journal of Number Theory* 3.03 (2007), pp. 349–354.

[2] Shyam Narayanan. "Modular Forms and Modular Congruences of the Partition Function". In: (2019).

Thank You

The slides are available at naufilsakran.com